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Mathematical Reviews

Vol. 20, No. 11 **DECEMBER, 1959** **Reviews 6974-7609**

LOGIC AND FOUNDATIONS

See also 7599, 7600.

6974:

Servien, Pius. La représentation mathématique des observables. Synthese **10**, 71–75.

This is a brief exposition of the author's works on the philosophy of physics, especially as they pertain to probability. The three most recent works referred to are: Recent communication to the International Colloquium on Physics, Paris, 1950; *Hasard et probabilités* [Presses Universitaires de France, 1949]; *Probabilités et quanta* [Hermann, Paris, 1948]. *L. J. Savage* (Rome)

6975:

Denk, F. Die mathematischen Begriffe und ihr Ausdruck. Synthese **10**, 173–180.

An operational point of view with respect to mathematical ideas, applied to pedagogy of mathematics.

H. Freudenthal (Utrecht)

6976:

Lorenzen, Paul. Ist Mathematik eine Sprache? Synthese **10**, 181–186.

According to the author, the basic function of language is the description of reality, whereas the basic function of mathematics is operational. In order to show the operational function of mathematics more clearly, the author recommends the development of a protologic which permits us to perceive logical activity "in statu nascendi". The separation of mathematics and language is made more difficult by the influence of the conception of mathematics as a description of real entities; conversely, the conception of mathematics as a form of language prevents us from understanding the intuitionistic rejection of the principle of the excluded middle.

E. W. Beth (Amsterdam)

6977:

Walter, Emil J. Die Ausbildung der mathematischen Zeichensprache und ihr Einfluss auf die Entwicklung der mathematischen Problemstellung. Synthese **10**, 187–189.

The author proposes, as a topic to be discussed by significs, the description of the domain of the operations which can be represented by a given symbolism; this will lead on to a comparative study of symbolisms providing a survey of the connections, identities, and divergencies of number systems, mathematical symbolisms, and so on.

E. W. Beth (Amsterdam)

6978:

Fitch, Frederic B. An extensional variety of extended basic logic. J. Symb. Logic **23** (1958), 13–21.

Fitch's extended basic logic K' [same J. **13** (1948), 95–106; MR **9**, 559] lacks any form of a principle of extensionality. In this paper K' is reformulated in such a way as to contain a "fairly strong" principle of extensionality to the effect that if 'a' and 'b' are of the same non-negative degree n, and if for all 'c_1', $\cdots$, 'c_n' the following conditions hold: $ac_1\cdots c_n \vdash bc_1\cdots c_n$, $bc_1\cdots c_n \vdash ac_1\cdots c_n$, $\sim(ac_1\cdots c_n)\vdash\sim(bc_1\cdots c_n)$, $\sim(bc_1\cdots c_n)\vdash\sim(ac_1\cdots c_n)$, then $0\vdash[a=b]$, where 0 is the empty class. The consistency of the resulting system is established by comparing it with systems varying slightly from K', especially in the rules governing '='.

R. M. Martin (Philadelphia, Pa.)

6979:

Schröter, Karl. Theorie des logischen Schliessens. II. Z. Math. Logik Grundlagen Math. **4** (1958), 10–65.

[For part I, see same Z. **1** (1955), 37–86; MR **17**, 814.] In this paper the author continues his systematic exposition of what he calls classical logic, namely two-valued logic applied to the usual truth functional connectives and the universal and existential quantifiers. The formalizations considered are (a) of the (usual) Hilbert style, (b) Gentzen's natural deduction and (c) Gentzen's calculus of sequents. The intended interpretations are naturally all based on more or less immediate variants of Bolzano's concept of consequence. The author gives austere proofs, with detailed distinctions, of the well-known results for each of these formalizations; particularly the deduction theorem, the cut theorem (when applicable), or the popular finiteness theorem. He notes some extensions of the completeness theorems for propositional logic and predicate logic which have not been mentioned in the literature although they are obtained by slight modification of the usual methods of proof. For example, the finiteness property for the simple theory of types over domains with a fixed (finite) number of individuals and of monadic predicate logic of second order. Also, there is a useful collection of remarks on systems for which a Skolem normal form theorem holds.

{Since this paper was published in 1958 and was intended to be a systematic exposition, it is necessary to observe that it reflects the state of classical logic before the war rather than at the present time. As far as the cut theorem is concerned the author's proofs may be simpler, as he himself stresses, than those of Gentzen's pioneer papers; but it is amazing that he neglects to use, and even to mention, the pellucid and precise presentation of Schütte [Math. Ann. **122** (1951), 47–65; MR **12**, 233; and particularly, Arch. Math. Logik Grundlagenforsch. **2** (1956), 55–67; MR **19**, 3]. The latter provides a very simple completeness proof of a cut-free formalization, and so, if one uses the concept of model and completeness, the elimination of cuts follows immediately. If, as is necessary, e.g., in studies of non-finite axiomatizability of certain formal systems, the cut theorem has to be proved in the system itself, then the former presentation is particularly useful. Concerning the other main result, namely, the finiteness (completeness) property, when there is particular stress on models for non-denumerably many axioms, Henkin's construction [J. Symb. Logic **14** (1949), 159–166; MR **11**, 487] of models out of the symbols of the system itself seems much more appropriate than the author's modification of Gödel's original proof which used Skolem normal forms; in fact, the latter is

really appropriate only if one wants to prove directly the completeness of Herbrand's cut-free rules for proving prenex formulae. This is an (apparently) stronger completeness result because Herbrand's rules are a proper subset of the usual one. As to coverage of the subject, we note that the author makes no mention of Herbrand type theorems ("théorème fondamental"); it is true that there is no adequate and neat formulation of their content in model theoretic terms, but the fact remains that they have become increasingly useful in foundations. Finally, Mostowski's discovery of generalized quantifiers which cannot be defined in terms of the universal and existential ones [Fund. Math. **44** (1957), 12–36; MR **19**, 724] shows that the author's restriction to classical logic (in his sense) is severe at the quantificational level; for, while it is known that all truth functions can be obtained by composition of the traditional ones, the very notion of a quantifier is quite problematic at the present time, and the problem of getting significant results about the more usual generalized quantifiers is wide open.}

G. Kreisel (Reading)

6980:

Mostowski, A. On a problem of W. Kinna and K. Wagner. Colloq. Math. **6** (1958), 207–208.

If M is a set and $\mathfrak{M}$ is the set of all subsets of M, then M is said to have property (E) if there exists a single-valued mapping φ of $\mathfrak{M}$ into itself such that, for every subset A of M containing at least two elements, $\varphi(A)$ is a nonempty proper subset of A. Consider the two propositions: (K) every set M has property (E); (O) every set can be ordered. According to Kinna and Wagner [Fund. Math. **41** (1955), 75–82; MR **17**, 950], (K) implies (O). The author proves that (K) and (O) are not equivalent on the basis of the usual axioms of set theory without the axiom of choice, by showing that (K) is false in a model $\mathfrak{B}^+$ of the axioms of set theory [see Mostowski, Fund. Math. **32** (1939), 201–252] in which (O) is true.

F. Bagemihl (Notre Dame, Ind.)

6981:

Bing, Kurt. On the axioms of order and succession. J. Symb. Logic **22** (1957), 141–144.

Consider the formula

$$a<b\rightarrow a'=b\vee a'<b$$

where a' is the successor of the natural number a. Hasenjaeger [Arch. Math. Logik Grundlagenforsch. **1** (1950), 30–31; MR **12**, 792] has shown that this formula is independent of the remaining axioms of the Hilbert-Bernays system (B) [Grundlagen der Mathematik, vol. 1, Springer, Berlin, 1934; p. 273] if '$0=0$' is replaced by '$a=a$'. Call this system (B_1). It consists of the following axioms: $a=a$, $a=b\rightarrow(A(a)\rightarrow A(b))$, $\sim(a<a)$, $a<b\&b<c\rightarrow a<c$, $a<a'$, $A(0)\&(x)(A(x)\rightarrow A(x'))\rightarrow A(a)$. Let (B′) and ($B_1'$) result from (B) and ($B_1$), respectively, by dropping this last axiom, the induction axiom. In the present paper it is shown that (B_1) is equivalent to (B) and that the axioms of (B_1) are independent. Further, the axioms of (B_1') as well as those of (B′) are provable from those of the Hilbert-Bernays system (A) [loc. cit., p. 263]. But the axioms of (A) are not provable from those of either (B′) or (B_1'), nor are (B′) and (B_1') equivalent.

R. M. Martin (Philadelphia, Pa.)

6982:

Goodstein, R. L. On the nature of mathematical systems. Dialectica **12** (1958), 296–316. (French summary)

The author's abstract of the paper is: "The crux of the dispute between formalism and intuitionism, it is held, is not whether certain entities exist or not, but how the term function shall be used in mathematics. The identification of effective definition with general recursion fails because an undefined function lies concealed beneath the requirement of a finite number of substitutions, and a fresh characterization of effective definition is sought in terms of a hierarchy of ordinal recursions. A correspondence exists between primitive recursive properies and direct proofs, of irrationality and transcendence, for instance, and between general recursive properties and indirect proofs. Mathematics is a concept creating activity and the distinction between a formal mathematics devoid of meaning, at one level, and a meaningful metamathematics at the next is considered to be untenable."

{The discussion of intuitionism is brought into question by two errors: formula (A) page 300 is only intuitionistically acceptable when '$(\forall x)P(x)$' is replaced by '$(\forall x)\neg\neg P(x)$'; and in the 5th and 6th lines from the bottom of page 306 and in line 1 of page 307, '$(\forall x)P(x)$' must be replaced by '$(\forall x)\neg P(x)$'.} *P. C. Gilmore* (Yorktown Heights, N.Y.)

SET THEORY

See also 7063.

6983:

Sunyer i Balaguer, F. Sur les types d'ordre distincts dont les n-ièmes puissances sont équivalentes. Fund. Math. **46** (1959), 221–224.

A, B étant deux ensembles ordonnés et α, β leurs types d'ordre on désigne par $\alpha\lesssim\beta$ le fait que A est isomorphe d'un sous-ensemble de B; les signes $\alpha<\beta$, $\alpha\approx\beta$ s'entendent d'eux-mêmes. On sait [cf. A. C. Davis, C. R. Acad. Sci. Paris **235** (1952), 924–926; MR **14**, 361] que la relation $\alpha^n=\beta^n$ n'implique pas nécessairement $\alpha=\beta$. Dans la Note on prouve que la relation $\alpha^n\approx\beta^n$ (et d'autant plus la relation $\alpha^n=\beta^n$) implique $\alpha\approx\beta$. La démonstration est basée sur le lemme suivant: $\alpha\beta\lesssim\gamma\delta\Rightarrow\alpha\lesssim\gamma\vee\beta\lesssim\delta$.

Đ. Kurepa (Zagreb)

6984:

Popruženko, J. Sur une proposition équivalente à l'hypothèse du continu. Colloq. Math. **6** (1958), 203–206.

Using standard methods from set theory, the hypothesis of the continuum is shown to be equivalent to the following proposition. There exists a subset H of E of type $\omega_c^*+\omega_c$ and having the following property. One can associate with each element p of E two elements of H, t and s, such that (1) p satisfies the condition $t\geqq p\geqq s$; (2) the set of elements x of H which satisfy the condition $t\geqq x\geqq s$ is at most denumerable.

Notation: E consists of all infinite sequences of positive integers which go to infinity. $s=(n_1, n_2, \cdots, n_i, \cdots)$ and $t=(m_1, m_2, \cdots, m_i, \cdots)$ being any two elements of E, write $s\geqq t$ if $\lim_{i\to\infty}(n_i/m_i)=\infty$.

S. Ginsburg (Hawthorne, Calif.)

COMBINATORIAL ANALYSIS

See 6995.

ORDER, LATTICES

See also 6983.

6985:

Wright, Fred B. The ideals in a factor. Ann. of Math. (2) **68** (1958), 475–483.

Let L be a complete orthocomplemented lattice such that whenever $a<b$ and a' is the orthogonal complement of a then $b=a\cup a'b$. Suppose also that in L there is defined an equivalence relation $\sim$ such that: (i) $a\sim 0$ only if $a=0$; (ii) $a_\alpha\sim b_\alpha$ implies $(\bigcup a_\alpha)\sim(\bigcup b_\alpha)$ provided each of $\{a_\alpha\}$, $\{b_\alpha\}$ is a pairwise orthogonal family; (iii) if $\{a_\alpha\}$ is a pairwise orthogonal family and $b\sim\bigcup a_\alpha$ then b possesses a decomposition $b=\bigcup b_\alpha$ with $\{b_\alpha\}$ pairwise orthogonal and $a_\alpha\sim b_\alpha$ for every α; (iv) $a\sim b$ does hold if a, b have a common complement.

Now call a subset I of L a p-ideal if (i) $a, b\in I$ imply $a\cup b\in I$; (ii) $a\in I$, $b\sim a_1\leq a$ together imply $b\in I$. The author proves: the p-ideals form a totally ordered, well ordered set if in L every pair of elements a, b can be compared, i.e., either $a\sim b_1\leq b$ or $b\sim a_1\leq a$.

I. Halperin (Kingston, Ont.)

6986:

Szász, G. Semi-complements and complements in semi-modular lattices. Publ. Math. Debrecen **5** (1958), 217–221.

If a lattice L has least element O, then $x\in L$ is called a proper semi-complement of $a\in L$ if $a\cap x=O$ and $x\neq O$; L is called semi-complemented if each element of L has a proper semi-complement. L is called semi-modular if $b\cap c<a<c<a\cup b$ implies the existence of t with $b\cap c<t\leq b$ and $(a\cup t)\cap c=a$; this is one of several definitions which reduce to the ordinary one if L has finite length. Let L be semi-modular and semi-complemented. Theorem 1: If for some $r\in L$ the set of all semi-complements of r has a maximal element m, then L has a greatest element and m is a complement of r. Theorem 2: If L has infinite length but every element other than I has finite dimension, then for each $r\in L$ other than O and I, and each non-negative integer k, there is a semi-complement x of r whose dimension is k.

P. M. Whitman (Baltimore, Md.)

6987:

Revuz, André. Topologies sur certains treillis complets. Arch. Math. **9** (1958), 342–346.

Dans cette Note l'auteur prouve le théorème suivant: La condition nécessaire et suffisante pour qu'un sous-treillis T d'un produit de chaînes C soit un espace $\tilde{\mathfrak{X}}$ (où $\tilde{\mathfrak{X}}$ est un espace topologique ordonné), l'ordre et la topologie induits par ceux de C, est que T soit un sous-treillis complet de C. C'est-à-dire que $\sup_T E$ et $\inf_T E$ existent pour toute partie E de T et que l'on ait $\sup_T E=\sup_C E$ et $\inf_T E=\inf_C E$.

A. T. Bharucha-Reid (Wrocław)

6988:

Matsushita, Shin-ichi. Zur Theorie der nichtkommutativen Verbände. I. Math. Ann. **137** (1959), 1–8.

The author develops in more detail, with proofs, some of his ideas on skew-lattices [C. R. Acad. Sci. Paris **236** (1953), 1525–1527; Proc. Japan Acad. **34** (1958), 407–410; MR **14**, 838; **20**#1640], especially the relationship between the definition in terms of an order relation and that in terms of postulates on the operations, with numerous forms of the absorption law. {In the corollary to theorem 5, it is necessary to include the assumption of associativity; for general treatment of this point, see Yu. I. Sorkin [Ukrain. Mat. Ž. **3** (1951), 85–97; MR **14**, 612].}

P. M. Whitman (Baltimore, Md.)

6989:

Grätzer, György; and Schmidt, Eligius. Ideals and congruence relations in lattices. I, II. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **7** (1957), 93–109, 417–434. (Hungarian)

The results of these papers are properly contained in those of the English paper reviewed below [#6990].

A. Kertész (Debrecen).

6990:

Grätzer, G.; and Schmidt, E. T. Ideals and congruence relations in lattices. Acta Math. Acad. Sci. Hungar. **9** (1958), 137–175.

The authors discuss many of the known results concerning the relationship between ideals and congruence relations in lattices. Some new proofs are given and some of the known theorems are generalized. Two new notions are introduced. A congruence relation θ is said to be separable if for each pair $a\leq b$ there exists a chain $a=z_0\leq z_1\leq\cdots\leq z_n=b$ such that θ is either the unit or null congruence relation on each z_i/z_{i-1}. A lattice L is said to be weakly modular if a/b weakly projective into c/d with a/b proper implies that there exists a proper subquotient c_1/d_1 of c/d such that c_1/d_1 is weakly projective into a/b. It is shown that the lattice of congruence relations on L form a Boolean algebra if and only if L is weakly modular and all congruence relations on L are separable. It is also shown that if L is a weakly atomic lattice with separable congruence relations, then the lattice of congruence relations on L is isomorphic to the weakly closed subsets of the partially ordered set of prime quotients. {Two corrections should be noted. The hypothesis $a\neq b$ is omitted in the statement of corollary 2 to lemma 6. The hypothesis of distributivity should be added to the statement of theorem 14.}

R. P. Dilworth (Pasadena, Calif.)

6991:

Grätzer, G.; and Schmidt, E. T. On the generalized Boolean algebra generated by a distributive lattice. Nederl. Akad. Wetensch. Proc. Ser. A. **61**=Indag. Math. **20** (1958), 547–553.

This paper contains a couple of proofs of the well-known result that any distributive lattice can be extended to a relatively complemented distributive lattice with zero element in such a way that the lattice of congruence relations is preserved.

R. P. Dilworth (Pasadena, Calif.)

6992:

Atsumi, Koichi. Notes on lattices. Proc. Japan Acad. **34** (1958), 510–512.

Following the notation of Y. Matsushima [Proc. Japan Acad. **32** (1956), 549–553; MR **18**, 713], characterizations of modular and distributive lattices are obtained. Let L be a lattice, $J(a, b)=\{x|\ x=(a\cap x)\cup(b\cap x)\}$, $CJ(a, b)$ defined dually. Necessary and sufficient for L to be modular is that $J(a, b)\cap XCCJ(a, b)$ for every $a, b\in L$, where $X=\{x|a\cap b\leq x\}$. Necessary and sufficient for L to be distributive is that $J(a, b)$ be an ideal for all $a, b\in L$; then $J(a, b)$ is the principal ideal generated by $a\cup b$. It follows that these relations [L. M. Blumenthal and D. O. Ellis, Duke Math. J. **16** (1949), 585–590; MR **11**, 369] are

pairwise equivalent if and only if L is modular:

$$(a\cap c)\cup(b\cap c)=c=(a\cup c)\cap(b\cup c);$$
$$(a\cap c)\cup(b\cap c)=c=c\cup(a\cap b);$$
$$(a\cup c)\cap(b\cup c)=c=c\cap(a\cup b).$$

[Reference might be made to other studies of these relations, e.g., M. Kolibiar, Mat.-Fyz. Časopis. Slovensk. Akad. Vied. **5** (1955), 162–171; MR **17**, 1177; E. Pitcher and M. F. Smiley, Trans. Amer. Math. Soc. **52** (1942), 95–114; MR **4**, 87; M. S. Gel'fand, Moskov. Gos. Ped. Inst. Uč. Zap. **71** (1953), 199–204; MR **17**, 704.]

P. M. Whitman (Baltimore, Md.)

6993:

Sussman, Irving. A generalization of Boolean rings. Math. Ann. **136** (1958), 326–338.

For an element a of a subdirect product R of domains (rings with identity and without proper zero divisors), let a^0 be the element of the full product such that $a_i^0=0$ if $a_i=0$ and $a_i^0=1$ if $a_i\neq 0$. a^0 is called the associate idempotent of a, and R is said to be an associate ring if $a^0\in R$ whenever $a\in R$. In an associate ring R, a^0 can be characterized by the conditions (i) a^0 is idempotent, (ii) $a^0a=aa^0=a$, and (iii) $a+1-a^0$ is not a zero divisor. A commutative ring with identity is an associate ring if and only if there exists for each $a\in R$ an element $a^0\in R$ satisfying (i)–(iii). Every regular ring without proper nilpotent elements is an associate ring.

A partial ordering is introduced by writing $a<b$ if $ab=a^2$, and two elements a and b are said to be compatible if $a^2b=ab^2$. In general, the relation of compatibility is not transitive. If a maximal set M of compatible elements in an associate ring R has a largest element, then $(M, <)$ is a distributive lattice, and M is a Boolean ring under the operations

$$a\dot{+}b=a+b-2a^0b,\quad a\dot{\times}b=a^0b.$$

All the Boolean rings obtained in this manner from R are isomorphic. *B. Jónsson* (Minneapolis, Minn.)

6994:

Moisil, Gr. C. Sur l'algèbre des relations binaires. I. Com. Acad. R. P. Romîne **8** (1958), 1251–1254. (Romanian. Russian and French summaries)

"L'auteur appelle algèbre de Schröder une algèbre de Boole avec une troisième loi de composition "|" ayant les propriétés $a|(b|c)=(a|b)|c$; $(a+b)|c=(a|c)+(b|c)$; $a|(b+c)=(a|b)+(a|c)$; il existe un élément neutre u tel que $a|u=u|a=a$; $0|a=a|0=0$.

Toute algèbre de Schröder finie est isomorphe à une algèbre de matrices dans une algèbre de Boole."

Résumé de l'auteur

GENERAL ALGEBRAIC SYSTEMS

6995:

Klein-Barmen, Fritz. Zur Theorie der Strukturen und Algebren. Math. Japon. **4** (1956), 83–94.

Let S be a subset of a Cartesian product $M_1\times M_2\times\cdots\times M_n$. Given $X_i\subseteq M_i$ for $i=1, 2, \cdots, n$, and $x_i\in X_i$ for $i=1, 2, \cdots, r<n$, the author is concerned with the number α of ways in which the remaining elements $x_i\in X_i$, $i=r+1, \cdots, n$ can be chosen so that $[x_1, x_2, \cdots, x_n]\in S$. If this number is independent of the particular choice of elements x_i, $i<r$, then the sequence of sets X_i is said to be of constant weight α. A typical theorem states that if X_i', X_i'' are of constant weights α and β, respectively, with $0<\alpha\leq\beta<\infty$, if $X_i'\cup X_i''$ is of constant weight β, and if the sets $X_i'\cap X_i''$, $i\leq r$ are non-empty, then the sequence of sets $X_i'\cap X_i''$ is of constant weight α.

B. Jónsson (Minneapolis, Minn.)

THEORY OF NUMBERS

See also 7062, 7121, 7172, 7173, 7247, 7397.

6996:

Jarden, Dov. Divisibility of V_{mn} by V_mV_n in the sequence associated with Fibonacci's sequence. Riveon Lematematika **12** (1958), 76–77. (Hebrew)

If V_n denotes the associated Fibonacci numbers, then $V_mV_n|V_{mn}$ if and only if m, n are relatively prime odd numbers. The corresponding theorem for Fibonacci numbers had been proved by the author in 1946 [Amer. Math. Monthly **53** (1946), 425–426; MR **8**, 313].

E. G. Straus (Los Angeles, Calif.)

6997:

van Yzeren, J. A note on an additive property of natural numbers. Amer. Math. Monthly **66** (1959), 53–54.

Omit every nth number from the sequence of natural numbers and form the sequence of partial sums. From this omit every $(n-1)$th number and form the sequence of partial sums, and so on. It was stated by Moessner [S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1951**, 29–30; MR **14**, 353], and proved by Perron and others, that the nth sequence is $1^n, 2^n, 3^n, \cdots$. Here this fact is exhibited as a consequence of applying Horner's process to express x^n as a polynomial in $x-i$, where $i=1, 2, 3, \cdots$.

H. Davenport (Cambridge, England)

6998:

Barnes, E. S. The construction of perfect and extreme forms. I. Acta Arith. **5** (1958), 57–79 (1959).

Let $f(\mathbf{x})$ be a positive definite quadratic form of determinant D and order n; let M be the minimum of $f(\mathbf{x})$ for integral non-zero vectors $\mathbf{x}$. Then $f(\mathbf{x})$ assumes the value M for a finite number of integral vectors $\mathbf{x}=\pm\mathbf{m}_1, \cdots, \pm\mathbf{m}_s$. The form $f(\mathbf{x})$ is said to be perfect if the s relations

$$F(\mathbf{m}_k)=M\quad(k=1, \cdots, s)$$

uniquely determine the coefficients of f. Perfect forms include all extreme forms, that is, those for which $M/D^{1/n}$ is a local maximum. Most known perfect and extreme forms are listed by Coxeter.

The author presents in this paper a method which yields many perfect forms with little labor starting with a known perfect or extreme form and producing a new form either by extending the range of values or by increasing the dimension of the known form. To this end he defines the lattice of a form f to be the lattice Tx, over integral vectors x, where $f=x'T'Tx$, and denotes this lattice by $\Lambda(f)$. A form f' is called a refinement of f if $\Lambda(f')$ contains $\Lambda(f)$. If Λ and Λ' are both n-dimensional, his basic theorem is: Let f' be a refinement of f with the same minimum M. Then if f is perfect, so is f'; and if f is extreme, so is f'.

The case when Λ has lower dimension than Λ' will be taken up in part II of this paper.

B. W. Jones (Boulder, Colo.)

6999:

Ledermann, Walter. An arithmetical property of quadratic forms. Comment. Math. Helv. 33 (1959), 34–37.

Let $\mathfrak{O}$ be the domain of rational numbers with odd denominators. For a in $\mathfrak{O}$, $a \equiv 0$ (2^α) will mean that the numerator of a is divisible by 2^α. The author proves, by induction on n, the following theorem: Let f be a quadratic form of degree n with coefficients and variables in $\mathfrak{O}$. If the determinant Δ of f is odd and if w is a vector such that $f(x, x) \equiv f(x, w)$ (mod 2) for all x, then $f(w, w) - \tau \equiv \Delta - \operatorname{sgn} \Delta$ (mod 4), where τ is the signature of f and $\operatorname{sgn} \Delta = +1$ or -1 according as $\Delta > 0$ or $\Delta < 0$.

A special case of the theorem where $\Delta = \pm 1$ was previously obtained as a corollary of topological investigations by Hirzebruch and Hopf [see #7272 below].

R. Ree (Vancouver, B.C.)

7000:

Cohen, Eckford. Generalizations of the Euler φ-function. Scripta Math. 23 (1957), 157–161 (1958).

Let J_k denote Jordan's totient: $J_k(n)$ is the number of ordered k-tuples of positive integers $<n$ whose greatest common divisor is relatively prime to n. Let φ_k denote the totient introduced earlier by the author: $\varphi_k(n)$ is the number of integers a for which $0 \leq a < n^k$, and (a, n^k) is not divisible by any kth power greater than 1. It is known that if n has the canonical factorization, $n = p_1^{\lambda_1} \cdots p_r^{\lambda_r}$, then

$$J_k(n) = n^k\left(1 - \frac{1}{p_1^k}\right) \cdots \left(1 - \frac{1}{p_r^k}\right) = \varphi_k(n).$$

In the present note the author supplies a new proof which makes clearer the relation of the functions J_k and φ_k to the Euler function $\varphi = J_1 = \varphi_1$.

V. L. Klee, Jr. (Copenhagen)

7001:

Kanold, Hans-Joachim. Über einen Satz von L. E. Dickson. III. Math. Ann. 137 (1959), 263–268.

[For parts I and II, see Math. Ann. **131** (1956), 167–179; **132** (1956), 246–255; MR **17**, 1185; **18**, 718.] Let k, r be positive integers, $\lambda > 1$ a real number, $V(n)$ the number of distinct prime divisors of n, and $\sigma_r(n)$ the sum of the rth powers of all positive divisors of n. The author determines a set of necessary and sufficient conditions for the existence of an infinity of integers n such that

$$\sigma_r(n) n^{-r} \geq \lambda > \sigma_r(d) d^{-r} \quad (d \mid n,\ d < n)$$

and $V(n) = k$. His paper generalizes previous results of H. N. Shapiro [Bull. Amer. Math. Soc. **55** (1949), 450–452; MR **10**, 514] and H. A. Bernhard [Proc. Amer. Math. Soc. **7** (1956), 469–471; MR **18**, 16]. The argument is similar to that used by the author in parts I and II.

L. Mirsky (Sheffield)

7002:

Jankowska, S. Les solutions du système d'équations $\varphi(x) = \varphi(y)$ et $\sigma(x) = \sigma(y)$ pour $x < y < 10000$. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 6 (1958), 541–543.

With the aid of J. W. L. Glaisher's "Number-divisor tables" [Cambridge Univ. Press, Cambridge, Eng., 1940; MR **2**, 33] the author finds all the solutions of the system of equations

$$\begin{cases} \phi(x) = \phi(y) \\ \sigma(x) = \sigma(y) \end{cases}$$

in natural numbers x and y, where $x < y < 10000$. Here $\phi(x)$ is Euler's totient function, $\sigma(x)$ is the sum of the divisors of x. There are 86 such solutions, but only 5 in which x is relatively prime to y. She poses the problems: I. Do there exist infinitely many pairs of integers a and b such that

$$(a, b) = 1,\ \phi(a) = \phi(b),\ \sigma(a) = \sigma(b),\ d(a) = d(b),$$

where $d(n)$ is the number of divisors of n? II. For every k does there exist a sequence of distinct integers $a_1, \cdots, a_k$ such that:

$$\phi(a_i) = \phi(a_j),\ \sigma(a_i) = \sigma(a_j) \text{ and } d(a_i) = d(a_j)$$

for all $1 \leq i < j \leq k$?

Both questions have been answered affirmatively by P. Erdös [see the following review].

S. Chowla (Boulder, Colo.)

7003:

Erdös, P. Solution of two problems of Jankowska. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 6 (1958), 545–547.

The author answers affirmatively two questions raised by Jankowska [see the preceding review]. The proof uses Brun's method [see also P. Erdös, Quart. J. Math. Oxford Ser. **6** (1935), 205–213; **7** (1936), 227–229; also the reviewer's note in Proc. Indian Acad. Sci., Sect. A **5** (1937), 37–39 and K. Prachar, Monatsh. Math. **59** (1955), 91–103; MR **16**, 904]. *S. Chowla* (Boulder, Colo.)

7004:

Erdös, P. On the distribution function of additive arithmetical functions and on some related problems. Rend. Sem. Mat. Fis. Milano 27 (1957), 45–49.

An additive arithmetical function $f(n)$ is one with the property that $f(mn) = f(m) + f(n)$ whenever m, n are relatively prime. The corresponding distribution function $\varphi(c)$ is the asymptotic density (if it exists) of those n for which $f(n) < c$. The paper gives a brief survey of known results, many of which are due to the author, and mentions several unsolved problems.

H. Davenport (Cambridge, England)

7005:

✱Knapowski, S. On the distribution of values of the Möbius function. Colloque sur la théorie des suites, tenu à Bruxelles du 18 au 20 décembre 1957, pp. 161–164. Centre Belge de Recherches Mathématiques. Librairie Gauthier-Villars, Paris; Établissements Ceuterick, Louvain; 1958. 167 pp. 220 fr. belges.

A full proof of the results announced here has appeared in Acta Arith. 4 (1958), 209–216 [MR **20** #3113].

N. G. de Bruijn (Amsterdam)

7006:

van der Corput, J. G. Some identities and inequalities involving maxima and minima. Nederl. Akad. Wetensch. Proc. Ser. A 61 = Indag. Math. 20 (1958), 239–251.

Let m and n denote positive integers. Let $Q = P_1 P_2 \cdots P_n$ denote a product of n distinct prime numbers $P_1, P_2, \cdots, P_n$; let $x_{\mu\nu}$ and $y_{\mu\nu}$ denote real numbers with

$$y_{\mu\nu} \leq x_{\mu\nu} \quad (1 \leq \mu \leq m;\ 1 \leq \nu \leq n);$$

and let $f(v_1, v_2, \cdots, v_m)$ be defined for all real values of $v_1, \cdots, v_m$. The author considers the sum

$$S = \sum_{D \mid Q} \mu(D) f(\min(z_{11}(D), \cdots, z_{1n}(D)), \cdots, \min(z_{m1}(D), \cdots, z_{mn}(D))),$$

where $\mu(D)$ is Möbius's function and

$$z_{\mu\nu}(D) = \begin{cases} x_{\mu\nu} \text{ for each } \nu \text{ with } P_\nu \nmid D, \\ y_{\mu\nu} \text{ for each } \nu \text{ with } P_\nu \mid D. \end{cases}$$

The author further defines: a divisor E of Q (divisor means positive divisor) possesses property Ω if for each positive integer $\nu\leqq n$ it is possible to find at least one positive integer μ such that $y_{\mu\nu}\leqq y_{\mu r}$ for each prime factor P_r of E.

The author proves the following identity

$$S=\sum_{E}\mu(E)f(\min(z_{11}(E),\cdots,z_{1n}(E)),\cdots,\min(z_{m1}(E),\cdots,z_{mn}(E))),$$

where the sum is extended over the divisors E of Q with property Ω. Thus the total contribution to S of the divisors D of Q without property Ω is zero.

S. Chowla (Boulder, Colo.)

7007:

Gupta, H. Partition of j-partite numbers into k summands. J. London Math. Soc. 33 (1958), 403–405.

In 1941, Erdös and the reviewer proved that $k!p_k(n)\sim\binom{n-1}{k-1}$ for $k=o(n^{\frac{1}{3}})$, where $p_k(n)$ is the number of partitions of n into exactly k summands [Duke Math. J. **8** (1941), 335–345; MR **3**, 69]. A year later the author gave a considerably simpler proof of the same result [Proc. Indian Acad. Sci. **16** (1942), 101–102; MR **4**, 190]. In the present paper he makes an analogue of his proof for j-partite numbers, i.e., j-dimensional vectors with rational integral components. The result is: for $n_1, n_2, \cdots, n_j\to\infty$ and $N_j=(n_1, n_2, \cdots, n_j)$, we have

$$k!p(N_j, k)\sim\prod_{i=1}^{j}\binom{n_i-1}{k-1},$$

where j is fixed, k is fixed, and $p(N_j, k)$ is the number of partitions of N_j into exactly k j-partite numbers, each of which has positive components.

J. Lehner (East Lansing, Mich.)

7008:

Volkmann, Bodo. Über Hausdorffsche Dimensionen von Mengen, die durch Ziffereigenschaften charakterisiert sind. VI. Math. Z. 68 (1958), 439–449.

[For part V, see same Z. **65** (1956), 389–413; MR **19**, 1161.] To the g-adic expansion of a number ρ with $0<\rho\leqq1$ the author associates the points

$$p_i(\rho)=\left(\frac{A_0(\rho, i)}{i},\cdots,\frac{A_{g-1}(\rho, i)}{i}\right)\ (i=1, 2, \cdots),$$

where $A_j(\rho, n)$ is the number of occurrences of the digit j among the first n digits in the expansion of ρ. The set of limit points of $\{p_i(\rho)\}$ is called $V(\rho)$.

The author obtains the following results on the set $V(\rho)$. 1. $(V\rho)$ is always a continuum contained in the set H in which the hyperplane $\sum x_j=1$ intersects the unit cube $0\leqq x_j\leqq1$. 2. Conversely, every continuum in H is $V(\rho)$ for some ρ. 3. Let $G(V)$ be the set of ρ for which $V(\rho)=V$ and let $d(x)=-\sum x_j\log x_j/\log g$ (with $0\log 0=0$). Then the Hausdorff dimension of $G(V)$ is given by $\dim G(V)=\min_{x\in V}d(x)$. 4. If T, S are arbitrary sets in H; $T\subseteq S$ and $G(T, S)$ the set of all ρ with $T\subseteq V(\rho)\subseteq S$; then $G(T, S)$ is empty if the minimal continuum T^* which contains T is not in S; $\dim G(T, S)=\min_{x\in T^*}d(x)$ if T is non-empty; and $\dim G(T, S)=\max_{x\in S}d(x)$ if T is empty.

E. G. Straus (Los Angeles, Calif.)

7009:

Godwin, H. J.; and Samet, P. A. A table of real cubic fields. J. London Math. Soc. 34 (1959), 108–110.

The authors construct a table of 830 real cubic fields of discriminant less than 20,000, extending a table of Delone and Fadeev [Trudy Mat. Inst. Steklov **11** (1940); MR **2**, 349] and a table for cyclic cubic fields of the reviewer and Gorn [J. Res. Nat. Bur. Standards **59** (1957), 155–168; MR **19**, 732]. This extremely valuable table is unaccountably omitted. (The Royal Society's Depository of Unpublished Mathematical Tables serves as memory-hole.)

If the field discriminant is Δ, then the Minkowskian-type result of Godwin [Proc. Cambridge Philos. Soc. **53** (1957), 1–4; MR **18**, 565] proves the existence of a field element α, which, together with its conjugates β and γ, satisfies $f(x)=x^3-ax^2+bx-c=0$ for integral coefficients with $S=[(\alpha-\beta)^2+(\beta-\gamma)^2+(\gamma-\alpha)^2]/2\leqq\Delta^{\frac{1}{2}}$. This relation easily limits the number of equations satisfied by $\alpha-[\alpha]$ and hence provides a finite method of enumerating all fields of discriminant less than Δ.

Equations with linear factors are rejected electronically. Next, if D is the root discriminant it must be decided if the square factor k^2 is removable. The case $k=2$ is determined electronically modulo 16 and the other prime k are treated by hand. A final hand calculation is made to determine if two fields of equal discriminant are distinct. If $y=p+qx+rx^2$ exists which satisfies $F(x)=y^3-Ay^2+By-C=0$ then the knowledge of $f(x)$ and $F(x)$ limits the number of possible p, q, r, by the discriminant values and by identities coming out of explicit substitution of the value of y.

The authors find no case of two fields with equal discriminant except for well-known cyclic cases. They state that Heilbronn supplied the case $D=130{,}397$, presumably the smallest non-cyclic positive discriminant with two fields, although the matter seems to be open.

For 49 fields $19{,}001<\Delta<20{,}000$ the class number, h, is evaluated by hand. Most of the steps in the process could be programmed for an electronic computer; compare the reviewer's work [J. Assoc. Comput. Mach. **2** (1955). 111–116; MR **16**, 866]. The results are that $h=1$ for 44 cases including $\Delta=19321=139^2$, the sole abelian case; while $h=2$ for $\Delta=19249$; $h=3$ for $\Delta=19220$, 19604, 19764; and $h=4$ for $\Delta=19821$. The coefficients are not given.

A table is provided for the frequency of occurrence of cyclic and non-cyclic type verifying results of Heilbronn (unpublished) and the reviewer [Proc. Amer. Math. Soc. **5** (1954), 476–477; MR **16**, 222]. *H. Cohn* (Tucson, Ariz.)

7010:

Vaidyanathaswamy, R. The algebra of cubic residues. J. Indian Math. Soc. (N.S.) 21 (1957), 57–66 (1958).

Let $p=6n+1$ be a prime and δ a primitive root mod p. Let G be the additive group of residue classes [0], [1], $\cdots$, $[p-1]$ mod p, $z=[0]$, $\pi_0=[1, \delta, \cdots, \delta^{6n-3}]$, $\pi_1=\delta\times\pi_0=[\delta, \delta^2, \cdots, \delta^{6n-2}]$, and $\pi_2=\delta^2\times\pi_0=[\delta^2, \delta^3, \cdots, \delta^{6n-1}]$. Let $\pi_i\cdot\pi_j$ denote the set of elements obtained by adding mod p each element of π_i to each element of π_j and write π_i^2 for $\pi_i\cdot\pi_i$. Then it is shown that

$$\pi_0^2=2nz+a_0\pi_0+a_1\pi_1+a_2\pi_2,$$

where a_0, a_1, a_2 are non-negative integers and $a_0+a_1+a_2=2n-1$. Similar expressions are obtained for $\pi_0\cdot\pi_1$, $\pi_0\cdot\pi_2$, etc., which involve the same constants a_0, a_1, a_2. These results are used to obtain others on the number of solutions of the cubic congruence $x^3+y^3\equiv1$ mod p $(=6n+1)$ and on the cubic character of the integers 2 and 3 mod p. *W. H. Simons* (Vancouver, B.C.)

7011:

Cugiani, Marco. Forme cubiche nei domini P-adici. Riv. Mat. Univ. Parma 8 (1957), 81-92.

Let α be a p-adic integer (p a rational prime). The author finds necessary and sufficient conditions that a given p-adic integer β can be represented in the form $x^3-\alpha y^3=\beta$, where x and y are integers. In particular, if $p\neq 3,7$, then a necessary and sufficient condition that $x^3-\alpha y^3$ represent all integers β is that α be the cube of a unit. The proof, which is entirely from first principles, is similar to that which he has already given in the quadratic case [Ann. Mat. Pura Appl. (4) **44** (1957), 1–22; MR **20** #1669]. *J. W. S. Cassels* (Cambridge, England)

7012:

David, Marcel. Contribution à l'étude algorithmique des approximations rationnelles simultanées de deux irrationnels. Application au cas cubique. Publ. Sci. Univ. Alger. Sér. A 3 (1956), 1-102.

"The author studies to what extent the classical results of the development of a quadratic irrational number into a continued fraction can be generalized to the characterization of two cubic irrationals in the same cubic field.

"A geometric study in two dimensions of this development of an irrational number provides several new results.

"The generalization in three dimensions for two irrationals, although not entirely satisfactory, enables one to connect and simplify the well-known reduction methods as well as the classical algorithmic theories. At the same time it gives several new results. Finally, an algorithm provides this characterization and enables one to obtain the corresponding units of the field." (From the author's summary) *J. Popken* (Amsterdam)

7013:

Schmidt, Wolfgang. Flächenapproximation beim Jacobialgoritämus. Math. Ann. **136** (1958), 365–374.

The notation is as in O. Perron, Math. Ann. **64** (1907), 1–76, where the Jacobi algorithm is studied in detail. Let (α_1, α_2) be a pair of real numbers, and let $(A_0^{(\nu)}, A_1^{(\nu)}, A_2^{(\nu)})$ be the corresponding triplets of integers given by the Jacobi algorithm. Put {misprint in text!}

$$D_\nu=$$

$$\left(\alpha_1-\frac{A_1^{(\nu)}}{A_0^{(\nu)}}\right)\left(\alpha_2-\frac{A_2^{(\nu-1)}}{A_0^{(\nu-1)}}\right)-\left(\alpha_1-\frac{A_1^{(\nu-1)}}{A_0^{(\nu-1)}}\right)\left(\alpha_2-\frac{A_2^{(\nu)}}{A_0^{(\nu)}}\right),$$

so that $D_\nu/2$ is the area of the triangle with vertices

$$(\alpha_1, \alpha_2),\ (A_1^{(\nu)}/A_0^{(\nu)},\ A_2^{(\nu)}/A_0^{(\nu)}),$$
$$(A_1^{(\nu-1)}/A_0^{(\nu-1)},\ A_2^{(\nu-1)}/A_0^{(\nu-1)}).$$

Denote by $\alpha=3.51\cdots$, $\beta=4.26\cdots$, ρ and σ the real roots of $\alpha^3-\alpha^2-31=0$, $\beta^3-3\beta^2-23=0$; $\rho^3-\rho^2-1=0$; $\sigma^3-\sigma-1=0$, respectively. The author proves the following result: For infinitely many ν,

$$D_\nu<\{\alpha A_0^{(\nu)}A_0^{(\nu-1)2}\}^{-1},$$

where α is best possible exactly when, in the algorithm, $a_1^{(\nu)}=0$, $a_2^{(\nu)}=1$ for $\nu\geqq N$; e.g., if $\alpha_1=\rho^2-\rho$, $\alpha_2=\rho$. If the algorithm does not end in this way, then for infinitely many ν,

$$D_\nu<\{\beta A_0^{(\nu)}A_0^{(\nu-1)2}\}^{-1},$$

where β is best possible exactly when $A_1^{(N+n)}=0$, $A_2^{(N+2n)}=1$, $A_2^{(N+2n+1)}=2$ for some N and all $n>0$; e.g., if $\alpha_1=\sigma^2-\sigma$, $\alpha_2=\sigma$. If the algorithm ends in neither of these two ways, then infinitely often

$$D_\nu<\{(13/3)A_0^{(\nu)}A_0^{(\nu-1)2}\}^{-1}.$$

The author suggests that this chain of alternatives can be continued and that similar formulae hold for the Jacobi algorithm of more than 2 numbers. *K. Mahler* (Manchester)

COMMUTATIVE RINGS AND ALGEBRAS

See also 7043.

7014:

Rayner, F. J. Relatively complete fields. Proc. Edinburgh Math. Soc. 11 (1958/59), 131-133.

D. S. Rim proved the following theorem [Duke Math. J. **24** (1957), 197–200; MR **19**, 244]: "Let k be a field with non-archimedean valuation v such that v can be extended uniquely to its algebraic closure. Then k is relatively complete, i.e., Hensel's lemma holds in k." The author proves the same statement where Hensel's lemma means the following stronger proposition: Let B be the valuation ring of k, P be the unique prime ideal of B. For $f(x)\in B[x]$, $\bar{f}(x)$ means the mod P reduced polynomial in $(B/P)[x]$. Then for any given triple of polynomials f, g, h in $B[x]$ such that (i) $\bar{f}=\bar{g}\bar{h}$, (ii) $(\bar{g}, \bar{h})=1$, (iii) $\Lambda g=1$, $\deg g>0$, (iv) $\deg h\leqq \deg f-\deg g$, there exists a pair of polynomials G, H in $B[x]$ such that $f=GH$, $\bar{g}=\bar{G}$, $\deg g=\deg G$, $\Lambda G=1$, $\bar{h}=\bar{H}$. Here Λg denotes the leading coefficient of g. *Y. Kawada* (Tokyo)

7015:

Kaplansky, I. Introduction to Galois theory. Notas Mat. No. 13 (1958), 153 pp. (Portuguese)

Lecture notes, by E. L. Lima, of a course at the University of Chicago, 1954. According to the introduction, the methods are inspired by Artin's book [*Galois theory*, Notre Dame Mathematical Lectures, no. 2, Univ. of Notre Dame, Notre Dame, Ind., 1942; MR **4**, 66].

7016:

Goldie, A. W. A note on the intersection theorem. J. London Math. Soc. 34 (1959), 47–48.

Let I, J, K be ideals in a Noether ring R. It is proved that if there exist a sequence $i_1, i_2, \cdots \in I$ and an $x\in R$ such that $x-i_nx\in(J+K)^n$ for all n, then there exists a sequence $p_1, p_2, \cdots \in I+J$ such that $x-p_nx\in K^n$ for all n. From this Krull's intersection theorem follows (and conversely). *T. Nakayama* (Nagoya)

7017:

Wallace, Andrew H. Analytic equivalence of algebroid curves. Canad. J. Math. 11 (1959), 1–17.

Soient k un corps algébriquement clos, R l'anneau de séries formelles $k[[X_1, \cdots, X_n]]$ et $\mathfrak{X}$ son idéal maximal. Un automorphisme T de R est dit d'ordre q ($\geqq 2$) si le minimum des ordres des séries $T(X_i)-X_i$ est q. Soient $\mathfrak{a}=(F_1, \cdots, F_r)$ et $\mathfrak{a}'=(F_1', \cdots, F_r')$ deux idéaux de R. On peut se demander si, lorsque les ordres des $F_i'-F_i$ sont suffisamment élevés, il existe un automorphisme T de R d'ordre élevé tel que $T(\mathfrak{a})=\mathfrak{a}'$ (en particulier si les variétés algébroïdes définies par les idéaux $\mathfrak{a}$ et $\mathfrak{a}'$ sont analytiquement équivalents). Le rapporteur avait montré que la réponse est affirmative lorsque $\mathfrak{a}$ et $\mathfrak{a}'$ sont princi-

paux et que l'origine est point singulier isolé de $\mathfrak{a}$ [J. Math. Pures Appl. 35 (1956), 1–6; MR 17, 788]. L'auteur montre ici que, si $r=n-1$, si $\mathfrak{a}$ est de hauteur $n-1$ (et est donc l'idéal d'une courbe algébroïde), et si les $F_i'-F_i$ sont d'ordres suffisamment élevés, il existe un automorphisme T de R d'ordre élevé qui applique les idéaux premiers de $\mathfrak{a}$ sur les idéaux premiers de $\mathfrak{a}'$. Il montre aussi que, si C et C' sont deux courbes algébroïdes qui sont analytiquement équivalentes par un automorphisme T d'ordre élevé, alors leurs projections "suffisamment générales" sont aussi analytiquement équivalentes. Les démonstrations sont assez délicates; elles se font par récurrence sur n au moyen de la méthode des "projections génériques". *P. Samuel* (Clermont-Ferrand)

7018:

Samuel, P. La notion de place dans un anneau. Bull. Soc. Math. France **85** (1957), 123–133.

A discussion of the possibility of generalizing the theorem on the extension of specializations to an arbitrary commutative ring R. Consider three conditions on the subring A of R: (1) A has a prime ideal which is not the contraction to A of any prime ideal in any larger subring of R; (2) the complement S of A in R is closed under multiplication; (3) if $P(X_1, \cdots, X_n) \in A[X_1, \cdots, X_n]$ has a term with leading coefficient 1 and maximal degree in each X_i, then $P(X)$ is not annulled by any n-tuple of elements of S. In general, the first condition implies the others, the third implies the second, and there are no other implications. If R is a field all three conditions are equivalent to A being a valuation ring of R, and this expresses the classical theory.

M. Rosenlicht (Evanston, Ill.)

7019:

Roquette, Peter. On the prolongation of valuations. Trans. Amer. Math. Soc. **88** (1958), 42–56.

Soit k un corps muni d'une valuation discrète $|\ldots|$ au sens de Krull, dont le rang (forcément fini) soit r. Soient $\Gamma_1 \subset \Gamma_2 \subset \cdots \subset \Gamma_r = \Gamma$ tous les sous-groupes isolés du groupe de valuation Γ de k. Comme on sait, à Γ_s correspond une valuation (de rang $r-s$) de k, qu'on obtient en remplaçant la valuation $|\alpha|$ de $\alpha \in k$ par 1 quand $|\alpha| \in \Gamma_s$ et en la laissant inchangée quand $|\alpha| \notin \Gamma_s$. Si l'on considère le corps résiduel de k par rapport à la valuation ainsi formée, la valuation initiale $|\ldots|$ y induit, d'une maniere évidente une valuation de rang s, et ce corps résiduel ainsi valué sera noté $\bar{k}^{(s)}$.

Soit K une extension de k d'un degré fini n. Alors, si $|\ldots|_1, |\ldots|_2, \cdots, |\ldots|_m$ sont tous les prolongements distincts de $|\ldots|$ à K, soient f_i, e_i le degré résiduel et l'ordre de ramification de K/k par rapport à $|\ldots|_i$. On a $\sum_i f_i e_i \leq n$. L'auteur montre qu'on a $\sum_i f_i e_i = n$ dans les deux cas suivants: (1) tous les $K^{(s)}/k^{(s)}$ sont séparables; (2) il existe dans k un corps de constantes κ tel que dim $(k/\kappa) = \dim(\bar{k}/\kappa) +$ rang de $|\ldots|$ (où $\bar{k} = \bar{k}^{(r)}$ est le corps résiduel de k par rapport à $|\ldots|$).

{Remarque du référent. Une grande partie de ce travail est consacré à la démonstration de $\sum_i f_i e_i \leq n$. Or, ce résultat est certainement connu depuis plusieurs années. En particulier, sa démonstration, faite par une généralisation immédiate de la méthode du cas $r=1$ (où l'on remplace le complété de k par une extension immédiate convenable) a été exposée, sans aucune prétention d'originalité, dans mon séminaire sur la théorie des corps valués, année 1953/54 (voir, par exemple, les exposées 14, 15bis et 15ter). La notion de corps complet par rapport à sa valuation a été, également, introduite dans le même séminaire (voir les exposés 1, 3, 15bis et 15ter de 1953/54) mais a certainement été employée plus tôt par d'autres mathématiciens. En particulier, I. Fleisher a indiqué, dans l'exposé 15bis (1953/54) de mon séminaire, que l'unicité de la topologie d'un espace vectoriel topologique de rang fini sur un corps valué topologiquement complet (quel que soit le rang de la valuation) et le fait que cet espace est complet sont les conséquences immédiates d'un résultat de Nachbin [Bull. Amer. Math. Soc. **55** (1949), 1128–1136; MR **11**, 368]. Les propositions équivalentes au lemme 2 et à la proposition 1 de l'auteur se trouvent dans l'exposé cité 15ter de mon séminaire de 1953/54. Toutefois, l'auteur est excusable de n'avoir pas connu le fait indiqué, car les résultats cités se trouvent disséminés soit dans les papiers inaccessibles au public, soit dans les travaux dont les valuations ne constituent pas le sujet central.} *M. Krasner* (Paris)

7020:

Auslander, Maurice; and Rosenberg, Alex. Dimension of ideals in polynomial rings. Canad. J. Math. **10** (1958), 287–293.

Let K be a commutative noetherian ring with identity element, and let $S=K[X_1, \cdots, X_n]$ be the polynomial ring in n indeterminates over K. If $\mathfrak{P}$ is a prime ideal of S, denote by $d(\mathfrak{P})$ the transcendence degree of the field of quotients of $S/\mathfrak{P}$ over the field of quotients of $K/\mathfrak{p}$, where $\mathfrak{p}=\mathfrak{P} \cap K$. The authors prove the following theorem: $d(\mathfrak{P})+r(\mathfrak{P})=n+r(\mathfrak{p})$, where $r(\mathfrak{P})$ and $r(\mathfrak{p})$ denote the ranks of $\mathfrak{P}$ and $\mathfrak{p}$ in S and K, respectively. This is a generalization of a well-known theorem when K is a field. Moreover, the latter classical theorem is proved here by homological methods.

If K is assumed to be a Dedekind ring, the authors obtain generalizations of the theorems on the dimension of intersection of two affine varieties. Theorem 1: Let $\mathfrak{P}_1$, $\mathfrak{P}_2$ be prime ideals in S such that $(\mathfrak{P}_1, \mathfrak{P}_2) \neq S$, and let $\mathfrak{J}$ be the ideal generated by $\mathfrak{P}_1 \otimes 1$ and $1 \otimes \mathfrak{P}_2$ in $S \otimes_K S$. Then if $\mathfrak{U}$ is a minimal prime ideal of $\mathfrak{J}$,

$$d(\mathfrak{U})=d(\mathfrak{P}_1)+d(\mathfrak{P}_2).$$

Theorem 2: Let $\mathfrak{P}_1$, $\mathfrak{P}_2$ be two prime ideals in S with $(\mathfrak{P}_1, \mathfrak{P}_2) \neq S$. Then if $\mathfrak{B}$ is any minimal prime ideal of $(\mathfrak{P}_1, \mathfrak{P}_2)$,

$$d(\mathfrak{B}) \geq d(\mathfrak{P}_1)+d(\mathfrak{P}_2)-n.$$

D. Buchsbaum (Providence, R.I.)

ALGEBRAIC GEOMETRY

7021:

★Lang, Serge. Introduction to algebraic geometry. Interscience Publishers, Inc., New York-London, 1958. xi+260 pp. $7.25.

This is, as the title shows, an introduction to algebraic geometry, covering the following topics: Places, algebraic sets, k-varieties, (affine, projective, abstract) varieties, Zariski topology, correspondences, product varieties, derived normal varieties, linear systems of divisors, differential forms, simple points, algebraic groups, Riemann-Roch theorem on curves.

This book does not cover the intersection theory, Grassman varieties, Chow varieties nor equivalence relations of cycles.

Chapter I contains some basic theorems on valuation rings (places). In chapter II, algebraic sets (in affine spaces), k-varieties (affine), homogeneous varieties (=affine cones; it seems to the reviewer that this use of the term "homogeneous variety" is not suitable because of the possibility of confusing it with the concept of homogeneous spaces) and product varieties (affine; over an algebraically closed field) are defined. In ch. III, the notion of varieties (=absolutely irreducible affine varieties) is defined and the Zariski topology is introduced. In ch. IV, correspondences (which may be called "geometric correspondences" in order to distinguish them from correspondences in the sense of cycles, which are not covered by this book) are treated and then the notion of abstract varieties is defined.

Ch. V concerns the normality of a variety, including Zariski's main theorem for birational correspondences, derived normal varieties, and projective normality. Divisors and linear systems of divisors are the topics of ch. VI. Ch. VII contains results due to Koizumi [J. Math. Soc. Japan **1** (1949), 273–280; MR **11**, 537] on differential forms. Ch. VIII concerns simple points and Ch. IX defines the notions of algebraic groups and abelian varieties.

The Riemann-Roch theorem on algebraic curves is proved in ch. X, where the fact that the sum of residues of a differential form on an algebraic curve is zero and Harnack's theorem (which asserts that the number of components of the real part of an irreducible curve is at most one more that its genus) are proved.

Each chapter, except for ch. VII, is accompanied by some comments and references to literature which are useful for students.

{The reviewer would like to add here that, although the author gave a new proof of Zariski's main theorem (pp. 124–130), there is an error on p. 128, which, the reviewer was informed, causes some students difficulty. The difficulty, however, could be removed by reducing first to the case where P is an algebraic point (by a ground field extension).} *M. Nagata* (Cambridge, Mass.)

7022:

Godeaux, Lucien. Sulle superficie algebriche di genere zero e di bigenere uno. Boll. Un. Mat. Ital. (3) **13** (1958), 531–534. (English summary)

In the present note we determine the algebraic surfaces of genera $p_a=p_g=0$, $P_2=1$ with a bicanonical curve of order greater than zero. *Author's summary*

7023:

Godeaux, Lucien. Sur les surfaces de genres nuls possédant des courbes bicanoniques irréductibles. C. R. Acad. Sci. Paris **248** (1959), 1764–1765.

7024:

Palman, Dominik. Flächen dritter Ordnung mit zwei absoluten Doppelpunkten, die den absoluten Kegelschnitt enthalten, und zirkuläre Kurven dritter Ordnung. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II. **13** (1958), 41–55. (Serbo-Croatian summary)

A cubic surface ϕ containing the absolute conic at infinity, and having two double points on this conic, contains also the real line p joining these. It has in general one plane of symmetry π, perpendicular to p and passing through the vertex O of the quadric cone enveloped by tangent planes to ϕ in points of the absolute conic. ϕ is thus generated by a pencil of circles in planes through p (i.e., planes perpendicular to π and parallel to the one real asymptote of c, the section of ϕ by π). These circles have as diameters the finite segments traced by c on lines parallel to its real asymptote. Conversely, given a circular plane cubic c, the circles thus defined generate a surface ϕ with the properties assumed above.

One circle of the pencil consists of p with a line d, perpendicular to π in D, the finite intersection of c with its real asymptote. There is thus a second pencil of circles on ϕ, with diameters in π, lying in planes through d.

p and d are the only real lines on ϕ. All the others are isotropic, and form (in pairs) four degenerate circles in the first pencil and three in the second. From this the author deduces the following property of the general circular cubic c: The intersection of any tangent to c parallel to its real asymptote with any tangent from the point D is equidistant from their two points of contact.

According as c is unipartite, bipartite, or nodal, so is ϕ. It is pointed out that ϕ cannot have a fourth node, if it is to be real, and contain the absolute conic at infinity, and two of the nodes are to be on this.

If the real point at infinity of c is an inflexion, d coincides with p and ϕ is a surface of revolution.

P. Du Val (London)

7025:

Rosina, Bellino Antonio. Sur le nombre des multilatères gauches d'ordre et genre donnés et sur la classification des courbes gauches algébriques. C. R. Acad. Sci. Paris **247** (1958), 1959–1961.

n lines in S_3, with a total of $n+p-1$ intersections (of pairs only), form a degenerate twisted curve of order n and genus p; this is called a skew multilateral of order n and genus p. The note outlines very sketchily an inductive method of finding all descriptively different types (i.e. having different intersection patterns) up to any given order, and indicates a proof that each of these is in fact a limiting form of an irreducible curve of order n and genus p. The author asserts that he has carried out the method for $n\leqq 7$ and has found in each known family of twisted curves of order $\leqq 7$ some members thus degenerating into lines. *P. Du Val* (London)

7026:

Brennan, J. G. A generalisation of a formula due to Schubert. Proc. Edinburgh Math. Soc. **11** (1958/59), 79–82.

Let b be an algebraic curve of genus g, on the complex ground field. Let there be given on b a linear series $g_m{}^r$ and an algebraic series $\gamma_n{}^1$ of index ν. A well-known formula by Schubert gives the number of groups of $r+1$ points which are common to a set of $g_m{}^r$ and a set of $\gamma_n{}^1$. In this note this result is generalised by finding the number of groups of s points which are common to a set of $g_m{}^r$ and a set of $\gamma_n{}^1$; these s points consist of a group of α_1 points of multiplicity $k_1, \cdots,$ of a group of α_p points of multiplicity k_p for the set of $g_m{}^r$ which contains them, with $r+1=\sum_{i=1}^{i=p}\alpha_i k_i$, and are all simple points of the set of $\gamma_n{}^1$ which contains them.

F. Gherardelli (Florence)

7027:

Gaeta, Federico. Sul calcolo effettivo della forma associata $F(W_{\alpha+\beta-n}{}^{gl})$ all'intersezione di due cicle effettivi puri $U_\alpha{}^g$, $V_\beta{}^l$ di S_n, in funzione delle $F(U_\alpha{}^g)$, $F(V_\beta{}^l)$ relative ai cicli secanti. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 269–276.

Questa nota fa parte di un seguito di lavori dello stesso A., in cui si tratte di problemi relativi all'intersezione di varietà algebriche proiettive pure (date su un campo k

di caratteristica zero), traendo abile profitto dai metodi del calcolo simbolico. (Risultante, Teorema di Bézout, ... sono fra i problemi più importanti trattati in questi lavori.)

Il risultato principale di questa nota è il seguente. Siano $U_\alpha{}^g$ e $V_\beta{}^l$ due cicli effettivi fini di dimensione α e β e di ordini g ed l rispettivamente, dello spazio proiettivo $P_n(k)$; siano $F(U)$ ed $F(V)$ le loro forme associate (zugeordnete Form): il calcolo effettivo della forma associata al ciclo intersezione di U e V, in funzione di $F(U)$ ed $F(V)$, si può ridurre al caso in cui uno dei cicli è lineare ($P=1$).
F. Gherardelli (Florence)

7028:

Marchionna, Ermanno. Sopra una relazione fra i generi di una superficie algebrica irregolare. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 23 (1957), 394–400.

È ben nota la disuguaglianza

(1) $p^{(1)}-1\leqq 8(p_a+1)+2(p_g+1)-\rho=10(p_a+1)+2g_1-\rho$

fra il genere lineare $p^{(1)}$ di una superficie algebrica, il suo genere aritmetico p_a, l'irregolarità g_1 (il genere geometrico p_g) e il numero base ρ. Per le superficie regolari ($g_1=0$), essendo sempre $\rho\geqq 1$, delle (1) si deduce un confine superiore per il genere lineare in funzione del solo genere aritmetico: $p^{(1)}-1\leqq 10(p_a+1)-1$. Il risultato principale di questa Nota consiste nel dimostrare che per le superficie algebriche irregolari ($g_1>0$), prive di fasci ellittici di curve, vale la disuguaglianza analoga $p^{(1)}-1<11(p_a+1)+5$. Essa si deduce subito dalla (1) quando sia $p_g\geqq 3g_1-6$, ma continua a valere per tutte qualle superficie algebriche che non contengono fasci ellittici di curve di genere p e per le quali non è simultaneamente: $p>1$, $p_a<2g_1-6$, $p_a>g_1-4$. *F. Gherardelli* (Florence)

7029:

Gemignani, Giuseppe. Sulle trasformazioni cremoniane che appartengono ad una reciprocità non degenere. Ann. Scuola Norm. Sup. Pisa (3) 12 (1958), 479–488.

If a Cremona transformation τ between two S_r is such that there is a non-singular correlation ω, such that whenever P and P' correspond in τ, P' lies in the hyperplane ω^P, then there are r linearly independent correlations with this property, and τ (and its inverse) is of order r. If $r=2$, and the correlation is not necessarily non-singular, the Cremona transformation is a de Jonquières transformation. *J. A. Todd* (Cambridge, England)

LINEAR ALGEBRA

See also 7046.

7030:

Spencer, A. J. M.; and Rivlin, R. S. The theory of matrix polynomials and its application to the mechanics of isotropic continua. Arch. Rational Mech. Anal. 2 (1958/59), 309–336.

The authors determine a basis for the invariants, under the orthogonal group, of a set of symmetric 3×3 matrices. The ideas which underlie the necessarily elaborate computations are the following. By means of the Cayley-Hamilton theorem it is shown that any product of symmetric 3×3 matrices can be expressed as a linear combination of matrix products of limited total degree, and of certain explicitly determined types, with coefficients expressible in terms of the traces of matrix products. From this result it is shown that a basis for the orthogonal invariants of five or fewer symmetric 3×3 matrices is given by the traces of an explicitly stated set of matrix products. The basis for the orthogonal invariants of a larger number of matrices is then deduced by using Peano's theorem. *J. A. Todd* (Cambridge, England)

7031:

Spencer, A. J. M.; and Rivlin, R. S. Finite integrity bases for five or fewer symmetric 3×3 matrices. Arch. Rational Mech. Anal. 2 (1958/59), 435–446.

The basis for the orthogonal invariants for five or fewer symmetric 3×3 matrices obtained in the paper reviewed above is reduced by obtaining certain identities between the members.

{Reviewer's note. It would be interesting to know whether the system finally obtained by the authors is irreducible. The authors do not appear to claim irreducibility.} *J. A. Todd* (Cambridge, England)

7032:

Goldman, A. J.; and Marcus, M. Convexity of the field of a linear transformation. Canad. Math. Bull. 2 (1959), 15–18.

Hausdorff's well-known theorem that the field of values of a linear transformation of unitary n-space is convex, is here proved as follows: By familiar methods, the general case is reduced to the 2-dimensional case, and this in turn to the case of matrices $\begin{pmatrix} a & 1 \\ 0 & 0 \end{pmatrix}$, $a\geqq 0$. This is disposed of by geometrical arguments in the complex plane. *C. Davis* (Providence, R.I.)

7033:

Marcus, M. On doubly stochastic transforms of a vector. Quart. J. Math. Oxford Ser. (2) 9 (1958), 74–80.

Let $\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_n)$ be an n-vector with different real coordinates. Let Ω_n be the set of $n\times n$ doubly stochastic matrices. Let $H_n(\lambda)=\Omega_n\lambda$. It is known that $H_n(\lambda)$ is the intersection of the half-spaces (1) $\sum_{j=1}^k t_{i_j}=\sum_{j=1}^k \lambda_j$ and (2) $\sum_{j=1}^n t_j=\sum_{j=1}^n \lambda_j$, where $1\leqq k<n$, $1\leqq i_1<\cdots<i_k\leqq n$ and $t\in H_n(\lambda)$. The paper is concerned with the structure of the points on the support planes (1) of $H_n(\lambda)$. A typical result follows. Let $t=S\lambda$ be on a support plane (1) with $S\in\Omega_n$. Then there exists an $n\times n$ permutation matrix P and $S_1\in\Omega_p$, $S_2\in\Omega_q$ with $p+q=n$ such that $S=P(S_1+S_2)$, where the second factor is the direct sum of S_1 and S_2. *S. Sherman* (Philadelphia, Pa.)

7034:

Erdös, P. Elementary divisors of normal matrices. IBM J. Res. Develop. 3 (1959), 197.

Elegant proof of the well known fact that the elementary divisors of a normal matrix ($N^*N=NN^*$) are all linear. As ground field is assumed a "generalized complex field", defined by extending an arbitrary ordered commutative field K with characteristic $\neq 2$ by adjoining a square root of a negative number.
H. Schwerdtfeger (Montreal, P. Q.)

7035:

Gross, Wolf. Sul calcolo del massimo modulo delle radici dell'equazione caratteristica di una matrice. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 24 (1958), 497–500.

Let M_n be the set of all n-square matrices with complex coefficients. For $A\in M_n$, let $\rho(A)$ be the maximum mo-

dulus of the eigenvalues of A. Let ϕ be any continuous, non-negative function of the elements of A, satisfying (i) $\phi(\gamma A)=\gamma\phi(A)$ for all $A\in M_n$ and all real $\gamma\geqq 0$, and (ii) $\phi(A)=0$ implies $A=0$. The author shows that there exists a constant λ, independent of A, such that $\rho(A)\leqq\lambda^{1/r}[\phi(A^r)]^{1/r}$, $r=1, 2, \cdots$. Also, $\rho(A)=\lim_{r\to\infty}[\phi(A^r)]^{1/r}$. There exists a constant ν, independent of A and B, such that $\phi(AB)\leqq\nu\phi(A)\phi(B)$. If (iii) $\phi(\alpha A)=|\alpha|\phi(A)$ for all complex α, then ν may be used for λ.

B. N. Moyls (Vancouver, B.C.)

7036:

Ostrowski, A. M. On the bounds of a one-parametric family of matrices. J. Reine Angew. Math. **200** (1958), 190–199.

If A is any real $n\times n$ matrix, it is known that orthogonal matrices U, V exist such that UAV is symmetric (in fact, diagonal). The problem of finding this diagonal form is equivalent to finding the roots of $B=AA^*$, or the polar decomposition of A. The present paper studies the special case in which A is in Jordan canonical form $\sigma E+H$; with this case settled, certain others are easy corollaries. Some of the results are:

$$\lambda(\sigma)=\sigma-\cos\frac{\pi}{n+1}+\sigma^{-1}\frac{n-1}{2n+2}\sin^2\frac{\pi}{n+1}+O(\sigma^{-2}),\ \sigma\to\infty;$$

$$\lambda(\sigma)/\sigma^n=1-\sigma^2-\tfrac{1}{2}(\beta_n+1)\sigma^4-\tfrac{1}{2}(\beta_n+\gamma_n+1)\sigma^6+O(\sigma^8),\ \sigma\downarrow 0;$$

$$\beta_2=-5,\ \beta_n=-1\ (n\geqq 3),\ \nu_2=14,\ \nu_3=-6,\ \nu_n=0\ (n\geqq 4);$$

$$\Lambda(\sigma)=\sigma+\cos\frac{\pi}{n+2}+\sigma^{-1}\frac{n-1}{2n+2}\sin^2\frac{\pi}{n+1}+O(\sigma^{-2}),\ \sigma\to\infty;$$

$$\Lambda(\sigma)=1+\sigma\cos\frac{\pi}{n}+\frac{n+2}{2n}\sigma^2\sin^2\frac{\pi}{n}+O(\sigma^3),\ \sigma\downarrow 0,$$

where λ_n is the minimum, Λ_n is the maximum root of AA^*. The paper J. Reine Angew. Math. **198** (1954), 143–160 [MR **16**, 558] is related to this one.

J. L. Brenner (Menlo Park, Calif.)

7037:

Amir-Moéz, Ali R.; and Horn, Alfred. Singular values of a matrix. Amer. Math. Monthly **65** (1958), 742–748.

The singular values (s.v.) ρ_i of an $n\times n$ matrix A of complex elements with eigenvalues (e.v.) λ_i are usually defined as the non-negative square roots of the e.v. of AA^*. These values are here called the absolute s.v. while the real s.v. α_i are defined to be the e.v. of $(A+A^*)/2$, the imaginary s.v. β_i the e.v. of $(A-A^*)/2i$. For A normal we have $\rho_i=|\lambda_i|$, $\alpha_i=\mathrm{R}(\lambda_i)$, $\beta_i=\mathrm{I}(\lambda_i)$. For a general A the connections between ρ_i and $|\lambda_i|$, α_i and $\mathrm{R}(\lambda_i)$, β_i and $\mathrm{I}(\lambda_i)$ are given by sets of inequalities which constitute necessary and sufficient conditions. Parts of these results have been obtained previously by A. Horn [Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 374–375; MR **13**, 565], Weyl [ibid. **35** (1949), 408–411; MR **11**, 37], Horn [Proc. Amer. Math. Soc. **5** (1954), 4–7; Amer. J. Math. **76** (1954), 620–630; MR **15**, 847; **16**, 105]. The relation between the λ_i and the e.v. γ_i of the unitary part of the polar decomposition of A is studied. A necessary condition is established; a set of necessary and sufficient conditions has since been obtained by Horn and Steinberg [see Pacific J. Math. **9** (1959), 541–550] and is given at the end of the paper. Further, $\sum\alpha_i^2+\sum\beta_i^2=\sum\gamma_i^2$ is established. For s.v. of sums and products previous results by Amir-Moéz [Duke Math. J. **23** (1956), 463–476; MR **18**, 105] are linked here with a remark of Wielandt. The relations between the e.v. of the unitary part of a product and the e.v. of the unitary parts of the factors are not completely cleared up so far.

O. Taussky-Todd (Pasadena, Calif.)

7038:

Bjerhammar, Arne. A generalized matrix algebra. Kungl. Tekn. Högsk. Handl. no. **124** (1958), 32 pp.

The study of pseudo-inverses for not necessarily square or non-singular matrices goes back to E. H. Moore [*General analysis*, Amer. Philos. Soc., Philadelphia, 1935] and was studied from an axiomatic point of view by von Neumann [Proc. Nat. Acad. Sci. U.S.A. **22** (1936), 707–713]. Recently several authors have taken it up again, since it is of importance in numerical work [see, e.g., Bjerhammar, Bull. Géodésique **1951**, 188–220; MR **13**, 312; Penrose, Proc. Cambridge Philos. Soc. **51** (1955), 406–413; **52** (1956), 17–19; MR **16**, 1082; **17**, 536; R. Rado, ibid. **52** (1956), 600–601; MR **18**, 371; M. P. Drazin, Amer. Math. Monthly **65** (1958), 506–514; MR **20** #5217]. Here a systematic study is made of various definitions of the inverse and its connection with the explicit solution of systems of not necessarily consistent linear equations. Corresponding to an inverse a unit matrix can be defined as the product of a matrix and its inverse. Numerical examples are given and statistical applications are made.

O. Taussky-Todd (Pasadena, Calif.)

7039:

Afriat, S. N. Analytic functions of finite dimensional linear transformations. Proc. Cambridge Philos. Soc. **55** (1959), 51–61.

This paper is a uniform expository account of the spectral theory of linear operators in a finite dimensional space, as it is developed with the aid of the operational calculus based on the formula

$$\phi(A)=(2\pi i)^{-1}\oint\phi(\lambda)(\lambda I-A)^{-1}d\lambda.$$

Reference is made to the modern Banach space work of Dunford, Lorch, and the reviewer, as well as to many earlier investigators of the algebraic theory for square matrices. There are some results relating to commuting matrices and the notion of $\phi(A)$ as a regular function of A. There is also consideration of $\phi(A)$ when ϕ is a multiple-valued analytic function.

A. E. Taylor (Los Angeles, Calif.)

ASSOCIATIVE RINGS AND ALGEBRAS

See also 6993, 7048, 7049, 7050a-b, 7099.

7040:

Kohls, C. W. The space of prime ideals of a ring. Fund. Math. **45** (1957), 17–27.

The Stone-topology, as generalized by Jacobson, and then McCoy, is examined for the space S of prime (as generalized by McCoy) ideals of a ring A. The relation of the generalized prime or primitive ideals of an ideal to the generalized prime, or primitive ideals, respectively, of the ring, is studied, with special attention to the routine of adjoining a unit.

R. Arens (Los Angeles, Calif.)

7041:

★Brauer, Richard. Some remarks on associative rings and algebras. Report of a conference on linear algebras, June, 1956, pp. 4–11. National Academy of Sciences-National Research Council, Washington, Publ. 502, v+60 pp. (1957).

Several loosely connected remarks on rings and

modules are given. The first is a precise, and generalized, formulation of the relationship between an m-ring (=ring with minimum condition) and what has sometimes been called its basic ring or core. Thus a ring T is called a reduced ring when $T/\mathrm{Rad}\,T$ is a direct sum of a finite number of skew fields, and it is shown that the problem of constructing all m-rings (with or without unit element) can be reduced to that of constructing all reduced m-rings. Two-sided ideals in an m-ring R are in 1-1 correspondence with those of the corresponding reduced ring T. Every automorphism of R is the product of an inner one and a certain automorphism derived from an automorphism of T. R is cleft if and only if T is so. Next, projective and injective modules are considered. One of the results is about the extension of isomorphisms of quotient modules (resp. submodules) of a projective (resp. injective) module, and its somewhat weaker form has been independently obtained by Morita and Tachikawa [Math. Z. 65 (1956), 414–428; MR 20 #1704; see also Morita, Kawada and Tachikawa, ibid. 68 (1957), 217–226; MR 20 #894]. The paper closes with remarks on algebras over a field, one of which shows how to reduce the construction of a general finite-dimensional indecomposable module of an algebra to that of special ones called "of maximal type" and then how to construct those special ones from indecomposable modules of smaller dimensions. *T. Nakayama* (Nagoya)

7042:

★**Baer, Reinhold. Meta ideals.** Report of a conference on linear algebras, June, 1956, pp. 33–52. National Academy of Sciences-National Research Council, Washington, Publ. 502, v+60 pp. (1957).

The author introduces a concept of meta ideal in analogy with Wielandt's notion of subnormal subgroup. An ideal chain of the ring R is a non-empty subset $\mathfrak{R}$ of subrings of R with the two properties: (a) If $S\neq R$ is a subring of R, then there exists T in $\mathfrak{R}$ such that $S\subset T$ (proper inclusion) and S is an ideal of the ring T; (b) If $\mathfrak{T}$ is a tower contained in $\mathfrak{R}$, then the join of $\mathfrak{T}$ belongs to $\mathfrak{R}$. Every member of an ideal chain of R is called a meta ideal of R. A meta ideal is of finite index if it is contained in a finite ideal chain. Some of the principal results follow. 1. Every subring of the ideal N of R is a meta ideal of R if N has the following property: If the ideal J of R is properly contained in N, then there exists an ideal T of R such that $TN+NT\subseteq J\subset T\subseteq N$. (Every nilpotent ideal N of R has this property; also, any ideal of R having this property is a nil ideal.) 2. Every idempotent meta ideal is an ideal. 3. The subring M of R is a meta ideal of finite index if and only if there exists an ideal J of R and a positive integer j such that $J^j\subseteq M\subseteq J$. 4. The ring R is isomorphic to the ring of integers modulo a square-free positive integer if and only if (a) the minimum condition is satisfied by the ideals in R, (b) the only nilpotent ideal of R is the zero ideal, and (c) every subring of R is a meta ideal of finite index in R. *W. E. Jenner* (Lewisburg, Pa.)

7043:

Drazin, M. P. Rings with central idempotent or nilpotent elements. Proc. Edinburgh Math. Soc. (2) 9 (1958), 157–165.

This is a leisurely exposition of various results, some new, for the rings of the title. Inasmuch as the proofs are (as the author remarks) "straightforward" and "elementary", a knowledge of "structure theory" is not a prerequisite for reading much of this paper. The first part contains three results not original with the author, although from all appearances the proof of one of these, theorem 2 (attributed to A. Rosenberg), is published here for the first time. This result is: If to each $x, y\in$ a ring R, there exists $g\in R$, depending on x and y, such that $x-x^2g$ commutes with y, then R is a CN-ring, that is, a ring with every nilpotent element central. (This is related to a theorem of I. N. Herstein [Canad. J. Math. 7 (1955), 411–412; MR 17, 121; theorem 3] which states that if g can be chosen to lie in the subring generated by x, then the above condition implies commutativity of R.) Reminiscent of results on semi-automorphisms, the concept "semi-π-regularity" is defined ($a^n=aga^n$, or $a^n=a^nga$, n, g, depending on a) and shown to coincide with "strong regularity" ($a=a^2g$) when nilpotent elements are absent. {Gertschikoff [Mat. Sb. (N.S.) 7 (1940), 591–597; MR 2, 121] proved that "no nilpotent elements" and "$a^n=a^{n+1}g$, n, g depending on a" imply strong regularity [quoted by Kaplansky in Trans. Amer. Math. Soc. 68 (1950), 62–75; MR 11, 317].} For π-regular rings, the converse of Rosenberg's theorem (quoted above) holds; in π-regular CN-rings, $a-a^2g$ is central, g depending on a. (For recent full discussions of "$a-a^2g$ is central" see Drazin [Rend. Circ. Mat. Palermo (2) 6 (1957), 51–64; MR 20 #3189], and Martindale [Proc. Amer. Math. Soc. 9 (1958), 714–721; MR 20 #4577].) Conditions in regular rings equivalent to "idempotents are central" are discussed. In the last section, conditions are given which imply the existence of a non-zero one-sided annihilator of a left ideal in an Artinian ring. {Comment: The author attributes a special case of theorem 1 to Herstein [Amer. J. Math. 75 (1953), 864–871; MR 15, 392; lemma 4]. This may mislead the unwary reader, inasmuch as the result in its full generality is cited by McLaughlin and Rosenberg [Proc. Amer. Math. Soc. 4 (1953), 203–212; MR 14, 718; p. 209] who refer to Herstein's previous work [Amer. J. Math. 73 (1951), 756–762; MR 13, 426; lemma 2]. While it is true that this latter result is not stated in its full generality, the author's proof is the same as the one given there.} *C. C. Faith* (University Park, Pa.)

7044:

Ivan, Ján. The radical and semisimplicity of direct product of algebras. Mat.-Fyz. Časopis. Slovensk. Akad. Vied 7 (1957), 158–167. (Slovak. Russian and English summaries)

Let A, A_1, A_2 be linear associative algebras of finite order over a field of characteristic 0 and let $A=A_1\times A_2$. From a unified point of view (based on the properties of the regular representation of the algebras) the following statements are proved: (1) A is nilpotent if and only if at least one of the algebras is nilpotent. A is semisimple if and only if both A_1 and A_2 are semisimple. (2) If R, R_1, R_2 are the radicals of A, A_1, A_2, respectively, then $R=(R_1\times A_2, A_1\times R_2)$ and $A/R=A_1/R_1\times A_2/R_2$. *Št. Schwarz* (Bratislava)

7045:

Harada, Manabu; and Kanzaki, Teruo. On Kronecker products of primitive algebras. J. Inst. Polytech. Osaka City Univ. Ser. A 9 (1958), 19–28.

Let A_i, $i=1, 2$, be left-primitive algebras over a field F and let D_i be the division algebra of commuting endomorphisms of a faithful simple left A_i-module. The authors investigate various questions concerning the structure of $A_1\otimes A_2$. There are many results of which the following are a representative sample. Theorem 1: If $D_1\otimes D_2$ is right primitive, then $A_1\otimes A_2$ is left primitive. If A_i is P.M.I., then $A_1\otimes A_2$ is primitive [P.M.I.] if and

only if $D_1 \otimes D_2$ is primitive [P.M.I.]. Theorem 3: Let C_i be the centre of D_i. If $A_1 \otimes A_2$ is P.M.I., then C_1 or C_2 is algebraic over F and $D_1 \otimes D_2$ is simple with minimum condition. Theorem 5: If C is algebraic over F, then $A \otimes C$ is (Jacobson) semi-simple if and only if $A \otimes A^*$ is semi-simple. In this case, $A \otimes B$ for any semi-simple B is semi-simple. *A. Rosenberg* (Evanston, Ill.)

7046:

Roşculeţ, Marcel N. Au sujet des formes extérieures. Acad. R. P. Romîne. Stud. Cerc. Mat. **9** (1958), 127–164. (Romanian. Russian and French summaries)

The author considers associative algebras (over the complex numbers) generated by elements $x^1, \cdots, x^n$ with defining relations (*) $x^ix^j+x^jx^i=A(x^i+x^j)+B$ for all i, j (A, B being fixed real numbers). On taking $A=B=0$ one obtains the exterior algebra; for these generalizations of the exterior algebra the author studies zero divisors and the radical. {Reviewer's note: on taking $y^i=x^i-\frac{1}{2}A$ one obtains (**) $y^iy^j+y^jy^i=\frac{1}{2}A^2+B$ for all i, j; thus the algebras considered are a certain type of Clifford algebras.} *B. Harris* (Evanston, Ill.)

NON-ASSOCIATIVE RINGS AND ALGEBRAS

7047:

Havel, Václav. Eine Bemerkung über die Semi-Homomorphismen der Alternativringe. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **8** (1958), 3–6. (Czech. Russian and German summaries)

Let σ be a mapping of an alternative ring A into an alternative ring B without divisors of zero. Introduce the following conditions: P_1: $(x+y)^\sigma=x^\sigma+y^\sigma$, $(xx)^\sigma=x^\sigma x^\sigma$, $(xyx)^\sigma=x^\sigma y^\sigma x^\sigma$ for all $x, y \in A$; P_2: $(x+y)^\sigma=x^\sigma+y^\sigma$, $(xy)^\sigma=x^\sigma y^\sigma$ for all $x, y \in A$; P_2': $(x+y)^\sigma=x^\sigma+y^\sigma$, $(xy)^\sigma=y^\sigma x^\sigma$ for all $x, y \in A$.

It is proved: Condition P_1 is equivalent to P_2 or P_2' if and only if $x^\sigma[y^\sigma(xy)^\sigma]=(x^\sigma y^\sigma)(xy)^\sigma$ for all $x, y \in A$. *Št. Schwarz* (Bratislava)

HOMOLOGICAL ALGEBRA

See also 7045.

7048:

Berstein, Israel. On the dimension of modules and algebras. IX. Direct limits. Nagoya Math. J. **13** (1958), 83–84.

[For parts I–VIII see S. Eilenberg, M. Ikeda and T. Nakayama, same J. **8** (1955), 49–57; MR **16**, 993; Eilenberg and Nakayama, ibid. **9** (1955), 1–16; MR **17**, 453; M. Auslander, ibid. **9** (1955), 67–77; MR **17**, 579; Eilenberg, H. Nagao and Nakayama, ibid. **10** (1956), 87–95; MR **18**, 9; Eilenberg and Nakayama, ibid. **11** (1957), 9–12; MR **19**, 118; Auslander, ibid. **11** (1956), 61–65; MR **19**, 14; J. P. Jans and Nakayama, ibid. **11** (1956), 67–76; MR **19**, 250; Eilenberg, A. Rosenberg and D. Zelinsky, ibid. **12** (1957), 71–93; MR **20** #5229.]

Let J be a directed set and let $\{\Lambda_j, \phi_{ij}\}$ be a direct system of rings with limit Λ. Let $\{A_j, \psi_{ij}\}$ be a direct system of abelian groups such that each A_j is a left Λ_j-module and $\psi_{ij}(\lambda a)=\phi_{ij}(\lambda)\psi_{ij}(a)$ for $\lambda \in \Lambda_j$, $a \in A_j$. Then the limit A of $\{A_j, \psi_{ij}\}$ is a left Λ-module.

The author then proves the following results. Theorem: If J is countable, then

$$\text{l. dim}_\Lambda A \leqq 1+\sup \text{l. dim}_{\Lambda_j} A_j.$$

Corollary 1: If J is countable, then

$$\text{l.gl.dim}\,\Lambda \leqq 1+\sup \text{l.gl.dim}\,\Lambda_j.$$

Corollary 2: If each Λ_j is an algebra over the commutative ring K_j, and $\{K_j, \nu_{ij}\}$ is a direct system (with ϕ_{ij} compatible with ν_{ij}), then Λ is a K-algebra, where K is the limit of $\{K_j, \nu_{ij}\}$. If J is countable, then

$$K\text{-dim}\,\Lambda \leqq 1+\sup_j K_j\text{-dim}\,\Lambda_j.$$

The proof of the theorem proceeds by induction on $n=\sup_j \text{l.dim}_{\Lambda_j} A_j$. *D. Buchsbaum* (Providence, R.I.)

7049:

Kaplansky, Irving. On the dimension of modules and algebras. X. A right hereditary ring which is not left hereditary. Nagoya Math. J. **13** (1958), 85–88.

A ring R is right [left] hereditary if every right [left] ideal in R is projective. The author shows that a ring may be right hereditary but not left hereditary by proving the following theorem. Let V be a vector space of countably infinite dimension over a field F. Let C be the algebra of all linear transformations on V with finite-dimensional range. Let B be the algebra obtained by adjoining a unit element to C and let $A=B\otimes_F B$. Then A is right hereditary but not left hereditary.

The proof proceeds by showing that A is a regular ring (i.e., for any a there is an x such that $axa=a$); that in a regular ring every countably generated (right) ideal is projective; and that in A every right ideal is countably generated. Thus A is right hereditary.

The author then produces a (non-countably generated) left ideal which is not projective.

D. Buchsbaum (Providence, R.I.)

7050a:

Hilton, P. J.; and Ledermann, W. Homology and ringoids. I. Proc. Cambridge Philos. Soc. **54** (1958), 152–167.

7050b:

Hilton, P. J.; and Ledermann, W. Homological ringoids. Colloq. Math. **6** (1958), 177–186.

In these two papers the authors take the objects out of categories and express the axioms for an abelian (or exact) category in terms of the maps. The object thus defined is called a homological ringoid. Precisely, a homological ringoid H is a set of elements together with two laws of composition (addition and multiplication) defined between some pairs of elements of H. Addition and multiplication are subjected to the same axioms that apply in a (noncommutative) ring when these operations are defined. Moreover, the existence of left and right identities for each element of H is postulated. Furthermore, H must satisfy the following additional axioms. I. (i) For every left-regular μ (i.e., $\alpha\mu=0$ implies $\alpha=0$) there is a right-regular ε (i.e., $\varepsilon\beta=0$ implies $\beta=0$) such that the right annihilator of μ (denoted by $R^0\mu$) is equal to the right ideal generated by ε (denoted by $R(\varepsilon)$); (ii) for every right-regular ε there exists a left-regular μ such that $L^0\varepsilon=L(\mu)$ (where L stands for left). II. (i) For every right-regular ε there exists a left-regular μ such

that $R(\varepsilon)=R^0\mu$; (ii) for every left-regular μ there exists a right-regular ε such that $L(\mu)=L^0\varepsilon$. III. Every α is expressible in the form $\alpha=\varepsilon\mu$, where ε is right-regular and μ is left-regular.

In the second paper, some applications are given, including a proof of the first Noether isomorphism theorem.

In the first paper, the authors consider a ringoid M which is obtained in a natural way from studying the category P of pairs (F, R) of free abelian groups F and subgroups R with fixed bases (i.e., presentations of finitely generated abelian groups). The pairs (F, R) can be represented by left-regular integral matrices A, and maps of these pairs can be represented as quadruples of integral matrices (A, T, S, B), where A and B are left-regular, and $AT=SB$. Addition of maps can be described by $(A, T_1, S_1, B)+(A, T_2, S_2, B)=(A, T_1+T_2, S_1+S_2, B)$ and multiplication by $(A, T, S, B)(B, V, U, C)=(A, TV, SU, C)$. If we agree to identify (A, T, S, B) with $(A, T+XB, S+AX, B)$ for all integral matrices X, and denote the equivalence class of (A, T, S, B) under this identification by $[A, T, S, B]$, the collection M of all $[A, T, S, B]$'s is a ringoid. The authors show that M is actually a homological ringoid and, by explicit computations in M, are able to compute the homology groups of finitely generated chain complexes and the induced homomorphisms of chain maps.

D. Buchsbaum (Providence, R.I.)

7051:

Heller, Alex. Homological algebra in abelian categories. Ann. of Math. (2) **68** (1958), 484–525.

Homological algebra is established upon the basis of abelian categories. Thus a category is called preadditive when, for every pair A, B of objects in it, the set $\mathrm{Hom}(A, B)$ of maps from A to B is made an abelian group by a bilinear composition. A preadditive category is additive when it possesses a "zero-object" and any two objects in it have a "direct sum". A map $f:A\to B$ in an additive category is a monomorphism when it induces a monomorphism $\mathrm{Hom}(C, A)\to\mathrm{Hom}(C, B)$ for every C. Epimorphism is defined dually. A short sequence $0\to A'\to A\to A''\to 0$ is a short exact sequence (s.e.s.) if the induced sequence $0\to\mathrm{Hom}(C, A')\to\mathrm{Hom}(C, A)\to\mathrm{Hom}(C, A'')\to 0$ is exact for every C. The exactness of a general sequence is also defined. An abelian category is an additive category with a subset of its maps, the set of "proper" maps, satisfying a certain set of axioms. According to one of the axioms every proper map has "kernel", "image" and "cokernel". An abelian category in which all maps are proper is an exact category; the notion coincides with Buchsbaum's [Trans. Amer. Math. Soc. **80** (1955), 1–34; MR **17**, 579], except for the axiom of direct sum. An additive category carries abelian structures, and, in fact, a minimal one derived from the set of splitting s.e.s., if and only if it satisfies the "cancellation axiom". For an abelian category K an abelian category K^S having proper s.e.s. in K as objects is defined. Also the category dK of "object-with-derivation" in an abelian category K is made an abelian category. Its subcategory d^PK, not abelian in general, is formed by objects with proper derivation. In d^PK the homology functor H is defined. On the other hand, on all objects-with-derivation, two kinds of cohomology, dual to each other, are defined. Also Ext and derived functors are introduced under the assumption of "enough projectives".

T. Nakayama (Nagoya)

GROUPS AND GENERALIZATIONS

7052:

Novikov, P. S.; und Adyan, S. I. Das Wortproblem für Halbgruppen mit einseitiger Kürzungsregel. Z. Math. Logik Grundlagen Math. **4** (1958), 66–88. (Russian. German summary)

Novikov's construction of a group with an unsoluble word problem [Trudy Mat. Inst. Steklov. no. **44**, Izdat. Akad. Nauk SSSR, Moscow, 1955; MR **17**, 706] was based on Turing's example of a cancellation semigroup with an unsoluble word problem [Ann. of Math. (2) **52** (1950), 491–505; MR **12**, 239]. However, Turing's paper is very condensed and requires a number of alterations to make all the arguments valid [see also W. W. Boone, Ann. of Math. (2) **67** (1958), 195–202; MR **19**, 1158]. The aim of the present paper is to show that the reference to Turing's cancellation semigroup can be avoided by going straight to Post's or Markov's examples of semigroups with unsoluble word problem. It is first proved that Novikov's $\mathfrak{B}$-systems [loc. cit., p. 96] which differ in their axioms from groups only in that $a^{-1}a$ may, but aa^{-1} may not, be inserted in words, can be interpreted as semigroups with one-sided (left) cancellation. Once this is established the authors show how Novikov's proof has to be modified to obviate any reference to two-sided cancellation semigroups. The central idea of the proof remains unaltered.

K. A. Hirsch (London)

7053:

Karrass, A.; and Solitar, D. Subgroup theorems in the theory of groups given by defining relations. Comm. Pure Appl. Math. **11** (1958), 547–571.

This paper deals with the general issue of finding the presentation of a subgroup H of group G whose presentation in terms of generators a_i and relations $R_\mu(a_i)=1$ is given. They place great stress on a "rewriting process" going from the form of a word $W(a_i)$ which is an element of H to the expression of this element in terms of the generators $s_k=J_k(a_i)$ of H.

Their process when applied to free groups shows that every Nielsen system of generators is (up to inverses) a Schreier system of generators [the converse result was obtained by the reviewer and T. Rado, Trans. Amer. Math. Soc. **64** (1948), 386–408; MR **10**, 98]. A result of a new kind is the following: If $H_1>H_2>\cdots>H_i>\cdots$ is a descending chain of subgroups of a free group whose intersection H is greater than 1, then every finitely generated free factor of H is a free factor of almost all the H_i. Several theorems give conditions that a subgroup H of a finitely presented group shall be a free group or free product.

Marshall Hall, Jr. (Columbus, Ohio)

7054:

Moran, S. Associative operations on groups. II. Proc. London Math. Soc. (3) **8** (1958), 548–568.

The author continues his study of associative operations on groups [same Proc. (3) **6** (1956), 581–596; MR **20**#3908] with a large number of results pertaining to verbal products of groups; he also defines a new associative operation. An interesting result is his generalization of the unique decomposition theorem for direct products of cyclic groups: Let $\prod^V A_\alpha$, $\alpha\in M$, be the verbal product of the groups A_α with respect to the word V, and likewise $\prod^U B_\beta$, $\beta\in N$, the verbal product of the

groups B_β with respect to the word U. If $\prod^V A_\alpha = \prod^U B_\beta$ and all A_α and B_β are infinite cyclic or cyclic primary groups, then for every A_α there exists a B_β isomorphic to it, and vice versa, and $\prod^V A_\alpha = \prod^U A_\alpha$ and $\prod^V B_\beta = \prod^U B_\beta$. The proof is simpler than the corresponding proof of Golovin for nilpotent products [Mat. Sb. N.S. **28(70)** (1951), 445–452; MR **13**, 105] and uses the circumstance that the result is true for direct products.

Some of the other results are: A verbal product of a set of groups cannot be decomposed into a free product except possibly the free product of two cyclic groups of order two. (This generalizes a theorem of Baer and Levi [Compositio Math. **3** (1936), 391–398].)

Conditions are given for: a) a verbal product of finite groups to be finite, and b) a verbal product to degenerate into a direct product. Using particular choices of verbal subgroups, soluble and nth Burnside products are defined using the derived series and nth powers, respectively. In contrast to nilpotent products [see Golovin, Mat. Sb. N.S. **27(69)** (1950), 427–454; MR **12**, 672], the soluble product of a finite number of finite groups need not be finite. The soluble product of locally infinite abelian groups is locally infinite. The famous Burnside problem can be rephrased: Is the nth Burnside product of a finite number of finite groups finite?

If F is the free product of the groups G_α, $\alpha \in M$, and $V(F)$ is a verbal subgroup of F, then the reduced verbal product $\prod^{\underline{V}} G_\alpha = F/V(F)$ is a new associative operation defined by the author. It is not regular [see Golovin, loc. cit.]. Other representations of reduced verbal products are given and used to investigate relations between the verbal and reduced verbal products of finite groups, periodic groups, locally finite groups, locally infinite groups, and p-groups.

The author announces the existence of other regular products which will be presented in subsequent papers.

R. R. Struik (Vancouver)

7055:

Erdős, Jenő. Torsion-free factor groups of free abelian groups and a classification of torsion-free abelian groups. Publ. Math. Debrecen **5** (1957), 172–184.

The following four rather surprising theorems are proved. If H is any subgroup of a free abelian group F with F/H torsion-free, then there exists a basis of F which is a complete system of representatives of the cosets of F modulo H if and only if $|F:H| = \text{rank } H$ ($|F:H|$ is the cardinality of F/H; a complete system of representatives gives precisely one element in each coset). Let F/H and F'/H' be isomorphic torsion-free factor groups of the free abelian groups F and F'. Then there exists an isomorphism φ of F onto F' with $\phi(H) = H'$ if and only if rank H $=$ rank H'. Given a homomorphism φ from a free abelian group F onto a direct sum of torsion-free groups G_α ($\alpha \in \Gamma$), then F is a direct sum of subgroups $F_\alpha (\alpha \in \Gamma)$ with $\varphi(F_\alpha) = G_\alpha$ for all $\alpha \in \Gamma$. Let H be any subgroup of an abelian group G with G/H torsion-free and nonzero. Then there exists a generating system of G which is a complete system of representatives of the cosets of G modulo H if and only if $|G:H| \geqq |H|$. The author then goes on to use these theorems to characterize torsion-free abelian groups in the following manner. If G is torsion-free of cardinality $\leqq m$, let G be represented as the factor group of a free abelian group F modulo a subgroup H with rank $F = \text{rank } H = m$. Let $\{b_\alpha\}$ and $\{b_\alpha'\}$ ($\alpha \in \Gamma$) be bases of F and H, respectively. Then to the group G the author associates the matrix $\|r_{\alpha\beta}\|$ ($\alpha, \beta \in \Gamma$) where $b_\alpha' = \sum_\beta r_{\alpha\beta} b_\beta$ and the $r_{\alpha\beta}$ are integers. He then shows easily that this is a one-one correspondence between torsion-free groups of cardinality $\leqq m$ and equivalence classes of row finite m by m matrices over the integers, where two matrices A and B are called equivalent if there exist regular matrices P and Q with $PAQ = B$ (all matrices are over the integers).

D. K. Harrison (Haverford, Pa.)

7056:

Papp, Zoltán. On the closure of the basic subgroup. Publ. Math. Debrecen **5** (1958), 256–260.

Let G be an abelian p-group (every element has p-power order). By the closure of G is meant the torsion subgroup of the completion of G with respect to the neighborhoods $p^n \cdot G$, $n = 1, 2, \cdots$. G is a pure subgroup of a group H if $(n \cdot H) \cap G = n \cdot G$ for all integers n. Kulikov has proved that a p-group G with $\bigcap p^n \cdot G = 0$ is a direct summand whenever it is a pure subgroup if and only if the reduced part of G is the closure of its basic subgroup (a pure subgroup L which is a direct sum of cyclic groups and has G/L divisible). In the present paper the author shows that the restriction $\bigcap p^n \cdot G = 0$ is unnecessary. His proof is somewhat simpler than the original proof of the weaker result.

D. K. Harrison (Haverford, Pa.)

7057:

Kovács, László. On subgroups of the basic subgroup. Publ. Math. Debrecen **5** (1958), 261–264.

Let G be an abelian p-group; i.e., every element in G has p-power order where p is a prime. A subgroup H in G has been called basic if H is a direct sum of cyclic groups and if both $n \cdot (G/H) = G/H$ and $(n \cdot G) \cap H = n \cdot H$ hold for every integer n. The author proves that a necessary and sufficient condition for a subgroup H of an abelian p-group G to be contained in a basic subgroup of G is that H be the union of an ascending sequence of subgroups of bounded height in G (L is called of bounded height in G if $(n \cdot G) \cap L = 0$ for some n). This is a generalization both of the existence of basic subgroups and of Kulikov's criterion that G is a direct sum of cyclic groups if and only if it is a union of an ascending sequence of subgroups of bounded height.

D. K. Harrison (Haverford, Pa.)

7058:

Fuchs, L. On the possibility of extending Hajós' theorem to infinite abelian groups. Publ. Math. Debrecen **5** (1958), 338–347.

Let G be an abelian group and S_i, $i = 1, \cdots, k$, be subsets such that every $g \in G$ can be written uniquely as $g = x_1 + \cdots + x_k$ with $x_i \in S_i$. Then G is a direct sum of the S_i. If each S_i is a cyclic subset (one of the form $\{0, a_i, \cdots, (n_i - 1)a_i\}$) and thus G is a finite group, then Hajós' theorem states that at least one of the S_i is a subgroup. The author gives the following two theorems which reduce in the finite case to Hajós' theorem and which express somewhat the extent to which Hajós' theorem can be generalized. If G contains no subgroups of type $Z(p^\infty)$, then G is a direct sum of arbitrarily many cyclic S_α ($\alpha \in J$) implies that one of the S_α is a subgroup if and only if $G = F \oplus \sum_m C(p)$, where F is a finite group, m is an arbitrary cardinal number, p is a fixed prime, and $C(p)$ is a cyclic group of order p. If any group G is a direct sum of finitely many weakly periodic subsets Q_i, $G = Q_1 + \cdots + Q_n$, then at least one of the Q_i is periodic. He calls a subset Q_i weakly periodic if there exists a $g \in G$, $g \neq 0$ such that any one of $g + Q_i$ and Q_i contains at most one element not in the other, while Q_i is called periodic if $g + Q_i = Q_i$ for some $g \neq 0$. These two theorems cover the two main opposite directions which generalizations of Hajós' theorem might run.

D. K. Harrison (Haverford, Pa.)

7059:

Hsiang, Wu-Chung. Abelian groups characterized by their independent subsets. Pacific J. Math. **8** (1958), 447–457.

The author defines an abelian group G as having: property (A) if every element is in a cyclic direct summand; property (B) if G is a direct sum of cyclic groups and for any subgroup H of G there exist bases $\{h_\alpha\}$ and $\{g_\beta\}$ of H and G such that for every h_α there exists a g_β with $h_\alpha \in (g_\beta)$ (i.e., there exists a basis of H which can be extended upwards to a basis of G); and property (C) if for every independent set $\{h_\alpha\}$ of G there exists an independent set $\{g_\alpha\}$ of G with $h_\alpha \in (g_\alpha)$ for all α and $(\{g_\alpha\})$ a direct summand (i.e., every independent set can be extended upwards to a direct sum of cyclic groups which is a direct summand). He characterizes these groups as follows: G has property A if and only if it is either of the form $\sum_p (\sum_{\alpha_p} Z_{\alpha_p}(p^{h_p}) \oplus \sum_{\beta_p} Z_{\beta_p}(p^{h_p+1}))$ or torsion-free with property A. It can easily be proved that these last are precisely the subgroups of the unrestricted direct sums of infinite cyclic groups. G has property B if and only if either G is free of finite rank or G is of the form

$$\sum_p (\sum_{\alpha_p} Z_{\alpha_p}(p^{h_p}) \oplus \sum_{\beta_p} Z_{\beta_p}(p^{h_p+1})).$$

G has property C if and only if G is either infinite cyclic or of the form $\sum_p (\sum_{\alpha_p} Z_{\alpha_p}(p^{h_p})$. All sums are direct.

D. K. Harrison (Haverford, Pa.)

7060:

Szele, Tibor. On a topology in endomorphism rings of abelian groups. Publ. Math. Debrecen **5** (1957), 1–4.

Let G be an abelian group and $E(G)$ be the ring of all homomorphisms of G into itself. If θ is a subset of $E(G)$, the author lets $E_\theta(G)$ be the ring of all elements of $E(G)$ which commute with all elements of θ, and he points out that any ring R with unit is isomorphic to some $E_\theta(G)$ (let G be the additive group of R and θ be all multiplications on the left by elements of R). A subset A of $E_\theta(G)$ is called closed if $f \in A$ whenever for every $x \in G$ there exists an infinite set $\{g_u\} \subset A$ with $f(x) = g_u(x)$ for all u. He shows that this makes $E_\theta(G)$ into a T_1 topological ring. An infinite set $\{\varphi_\nu\} \subset E_\theta(G)$ converges to $\varphi \in E_\theta(G)$ if every open set B containing zero contains all but a finite number of the $\varphi - \varphi_\nu$, and a Cauchy system is defined accordingly. It is shown that $E_\theta(G)$ is always complete. An appropriate definition of summable is given and it is shown that an infinite set is summable if and only if it converges to 0.

D. K. Harrison (Haverford, Pa.)

7061:

Zappa, Guido. Sugli automorfismi uniformi nei gruppi di Hirsch. Ricerche Mat. **7** (1958), 3–13.

An automorphism ϕ of the group G is "without fixed elements" (except the neutral element) if the mapping $-1+\phi$ is one-to-one; the author calls ϕ uniform if $-1+\phi$ is onto G. For finite groups these two properties are evidently equivalent; but not for infinite groups, as is seen from the example of the infinite cyclic group and the automorphism that inverts its elements. The author first gives a new proof of the theorem of Witt, Itô and Higman [G. Higman, J. London Math. Soc. **32** (1957), 321–334; MR **19**, 633] that if a finite soluble group G admits an automorphism ϕ of prime order without fixed elements, then G is nilpotent. He then extends this result to infinite soluble groups with maximal condition (also known as "polycyclic groups" or "gruppi di Hirsch"), under the further assumption that ϕ is also uniform. Finally he shows by the example of the infinite dihedral group and an automorphism of order 2 without fixed elements that the assumption of uniformity is needed.

B. H. Neumann (Manchester)

7062:

Yacoub, K. R. On the existence of non-linear semi-special permutations on $[p^\alpha q]$. Nederl. Akad. Wetensch. Proc. Ser. A. **61** = Indag. Math. **20** (1958), 564–572.

The author shows that there always exist, for $\alpha > 1$, non-linear semi-special permutations on the cyclic groups of the title. For terminology and background, see the author's papers [Proc. Glasgow Math. Assoc. **2** (1955), 116–123; **3** (1958), 164–169; Publ. Math. Debrecen **5** (1958), 246–255; MR **17**, 11; **20** #5235c, #5235a].

F. Haimo (St. Louis, Mo.)

7063:

Gilbert, Jimmie D. Groups which induce a partition of a set. Amer. Math. Monthly **66** (1959), 121–123.

On sait qu'à tout groupe de permutations d'un ensemble, E, correspond une équivalence (de transitivité) dans laquelle deux éléments de E appartiennent à la même classe s'il existe une permutation du groupe appliquant le premier sur le second. D'ailleurs, à chaque partition de E correspond un tel groupe (de transitivité ou d'inertie), produit des groupes symétriques définis par chaque classe; et le treillis de ces groupes d'inertie est isomorphe à celui des partitions de E [G. Birkhoff, Proc. Cambridge Philos. Soc. **31** (1935), 433–454; 449, th. 22]. D'autre part Cauchy [Oeuvres, Sér. 1, vol. 9, Gauthier-Villars, Paris, 1896; p. 296] a établi que, si les sous-groupes transitifs d'un groupe G, qui laissent invariantes les classes de transitivité de G, sont d'ordre $r_1, r_2, \cdots$, resp., l'ordre de G est divisible par le P.P.C.M. de $r_1, r_2, \cdots$. Le groupe de transitivité d'une partition n'est pas le seul qui respecte chaque classe de l'équivalence individuellement; enfin le groupe d'inertie ne doit pas être confondu avec le groupe d'automorphisme ou de décomposition de l'équivalence, lequel contient des permutations pouvant projeter les unes sur les autres les classes d'égale puissance de la partition. Quand un groupe G définit une partition P, on dit que P est "induite" par G. Th. I: Si l'ensemble fini, E, est décomposé en classes M_i, possédant m_i éléments resp., et si r est le P.P.C.M. des m_i, alors l'ordre k de tout groupe G, "induisant" cette partition, est divisible par r. Th. 2: Si de plus G est abélien, k est au plus égal au produit des m_i. Th. 3: Si de plus les m_i sont premiers entre eux deux à deux, G est le produit direct de sous-groupes qui "induisent" respectivement chaque classe de la partition.

A. Sade (Marseille)

7064:

Brauer, Richard; and Reynolds, W. F. On a problem of E. Artin. Ann. of Math. (2) **68** (1958), 713–720.

Let G be a finite group whose order is divisible by a prime p but not by p^2, and assume that G is its own commutator subgroup. Then the number of Sylow p-groups of G is $1+rp$ where r is of the form $r=(hup+u^2+u+h)/(u+1)$ with positive integers u and h, except when one of the following groups is a homomorphic image of G: (i) $L_2(p)$ with $p>3$ (then $r=1$); (ii) $L_2(p-1)$, where $p>3$ is a Fermat prime (then $r=(p-3)/2$). Here $L_2(m)$ denotes the group of unimodular linear collineations of a Desarguean projective line which belongs to a finite field with m elements. The following result answers a question raised by Artin: If the order g of a simple, non-cyclic group G has a prime factor $p>g^{1/3}$, then G is isomorphic either to (i) or to (ii).

L. Fuchs (Budapest)

7065:
Ferschl, Franz; und Nöbauer, Wilfried. Halbordnungen von endlichen Gruppen. Arch. Math. 9 (1958), 401–406.

Let F be the class of all finite groups, and define two elements of F to be equivalent if they are isomorphic. The set Γ of all equivalence classes of F is partially ordered by defining $A \leqq B$ if a member of A is a subgroup of a member of B. If F is restricted to abelian groups, then the resulting Γ is a distributive lattice. In general, for each A in Γ, let $\Gamma(A)=\{X \in \Gamma : X \leqq A\}$. The authors determine all A in Γ for which $\Gamma(A)$ is a chain; namely, the cyclic p-groups, the elementary abelian p-groups, the non-abelian group of order q^3 all of whose elements are of order q (for each prime $q>2$) and the quaternion group. Finally, all A in Γ are determined for which $\Gamma(A)$ contains 6 or less elements.

P. F. Conrad (New Orleans, La.)

7066:
Anonymous. Correspondence. Ann. of Math. (2) 69 (1959), 247–251.

L'auteur de cette note, qui signe "R. Lipschitz", attire l'attention sur le fait que, dans le travail de ce dernier *Untersuchungen über die Summen von Quadraten* [M. Cohen, Bonn, 1886], on trouve: (1) pour toute matrice orthogonale U, une formule explicite donnant en fonction des éléments de U, l'élément s de l'algèbre de Clifford (défini à un facteur près) tel que $x \to sxs^{-1}$ soit la transformation orthogonale de matrice U; (2) la définition de la norme spinorielle. En outre, ces expressions sont mises en relation avec la représentation de Cayley de U. {On peut signaler ici que l'ouvrage de Lipschitz est absent de nombreuses bibliothèques, et qu'il est à regretter que les mathématiciens allemands n'aient pas jugé bon de publier les oeuvres complètes de ce mathématicien (ni d'ailleurs celles de Kummer et de Frobenius, ce qui est encore plus regrettable).}

J. Dieudonné (Paris)

7067:
Ohara, Akiko. La structure du groupe des similitudes directes $GO_6^+(Q)$ sur un corps de caractéristique 2. Osaka Math. J. 10 (1958), 239–257.

A detailed account of the isomorphisms between 6-dimensional orthogonal groups over fields of characteristic not 2 and classical groups of lower dimension has been given by J. Dieudonné [Acta Math. 87 (1952), 175–242; MR 14, 239]. The author extends this work to characteristic 2. The precise group studied is the group of direct similitudes, $GO_6^+(Q)$, defined as follows. Let K be a field of characteristic 2, E a 6-dimensional vector space over K, $Q(x)$ a non-degenerate quadratic form on E, $f(x, y)\equiv Q(x+y)+Q(x)+Q(y)$. The nonsingular linear transformations u such that $Q(u(x))\equiv r_u Q(x) (r_u \in K)$ form the group of similitudes, $GO_6(Q)$. If v is a non-singular linear transformation such that $f(v(x), v(y))\equiv r_v f(x, y) (r_v \in K)$, the pseudo-discriminants $\Delta(Q)$, $\Delta(Q_1)$ of $Q(x)$ and $Q_1(x)\equiv Q(v(x))$ are related by an equation of the form $\Delta(Q_1)= r_v^2\Delta(Q)+D(v)^2+r_v D(v)$. Then $v \in GO_6^+(Q)$ if and only if $v \in GO_6(Q)$ and $D(v)=0$. The author's results and methods closely resemble Dieudonné's.

G. E. Wall (Sydney)

7068:
Ore, Oystein. On coset representatives in groups. Proc. Amer. Math. Soc. 9 (1958), 665–670.

The author studies the following question. Given a group G and two subgroups H and K, under what circumstances does there exist a set S of elements of G which are simultaneously representatives of right cosets of H and left cosets of K? (A) $G=\sum gH=\sum Kg$, $g \in S$. The case $K=H$ is of special interest. He observes that it is necessary and sufficient that such a representation exist for every double coset $D=KdH$. A number of conditions are found. For $d \in G$ write

$$D_H=H \cap d^{-1}Kd,\ D_K=K \cap dHd^{-1}.$$

First: A necessary and sufficient condition for the existence of common representatives is that for every $d \in G$ the indices $[H:D_H]$ and $[K:D_K]$ have the same cardinal number. If for any subgroup A we write $I_d(A)= \bigcap d^{-i}Ad^i$, $i=-\infty$ to $+\infty$, various conditions depend on this group. Thus, a necessary condition for the existence of common representatives is that $[H:I_d(H\cap K)]= [K:I_d(H \cap K)]$. If all indices $[K:I_d(H)]$ and $[K:I_d(K)]$ are finite, a sufficient condition is that $[H:H\cap K]= [K:H\cap K]$.

The results generalize the earlier known sufficient conditions, namely, that H and K be of the same finite order or of the same finite index.

Marshall Hall, Jr. (Columbus, Ohio)

7069:
Fuchs, L. Note on ordered groups and rings. Fund. Math. 46 (1959), 167–174.

This paper develops necessary and sufficient conditions for a partial ordering of a group or ring to be extendable to a linear order. From these, most of the known conditions for a group or ring to admit a linear order are obtained as easy corollaries. The question of when a partial order can be represented as an intersection (or conjunction) of linear orderings is also treated.

R. S. Pierce (Seattle, Wash.)

7070:
Clifford, A. H. Totally ordered commutative semigroups. Bull. Amer. Math. Soc. 64 (1958), 305–316.

This article is a thorough survey of the theory of ordered semigroups, written by a leading worker in the field. The first part of the article deals with the algebraic theory, and the second part with the topological theory.

A. Shields (Ann Arbor, Mich.)

7071:
Thierrin, Gabriel. Sur la structure des demi-groupes. Publ. Sci. Univ. Alger. Sér. A 3 (1956), 161–171.

The paper consists of three only loosely connected parts.

In the first section the decomposition of four types of semigroups into a subdirect product of semigroups of a known structure is given. The types of semigroups treated are: Abelian torsion semigroups, the infinite cyclic semigroup, semigroups S in which for every $a \in S$ there is an integer $n>1$ such that $a \in S-S^n$, the semigroups in which every pair of idempotents commutes and $a \in a^2S \cap Sa^2$ holds for every $a \in S$.

An equivalence relation R is right regular if $aRb \to axRbx$; it is left simplifiable if $axRbx \to aRb$. In the second part of the paper necessary and sufficient conditions that a right simplifiable equivalence R be right regular for some types of semigroups are given.

A semigroup is called "inversible" if for every $x \in S$ there is a $y \in S$ such that xy is an idempotent. A semigroup is called "rectangular" if $ax=bx=ay=m$ implies $m=by$. In the third part of the paper an isomorphic representation of any semigroup which is at the same time inversible and rectangular by means of certain quadruples is given.

Št. Schwarz (Bratislava)

7072:

Kolibiarová, Blanka. On the semigroups, every subsemigroup of which has a left unit element. Mat.-Fyz. Časopis. Slovensk. Akad. Vied 7 (1957), 177–182. (Slovak. Russian and English summaries)

Let S be a semigroup and $J(S)$ the set of all idempotents $\in S$. An L-semigroup is a semigroup in which each subsemigroup contains a left unit element. A semigroup S is an L-semigroup if and only if S is a class sum of disjoint torsion groups, and $J(S)$ is an L-semigroup.

The structure of right ideals of an L-semigroup is studied. In fact, every right ideal of an L-semigroup S is at the same time a two-sided ideal of S. (An analogous statement for left ideals does not hold.)

The problems discussed are generalizations of or related to some problems treated by N. N. Vorob'ev, [Dokl. Akad. Nauk SSSR **88** (1953), 393–396; MR **14**, 718].

Št. Schwarz (Bratislava)

7073:

Ivan, Ján. On the matrix representations of simple semigroups. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **8** (1958), 27–39. (Slovak. Russian and English summaries)

Let S be a simple semigroup without zero in which the product of two idempotents is an idempotent. Let $A_K(S)$ be the corresponding semigroup algebra over a field K of characteristic 0. New proofs of known results concerning $A_K(S)$ are given. In particular: $A_K(S)$ is semisimple if and only if S is a group; if G is the group component of S and N the radical of $A_K(S)$, then $A_K(S)/N \cong A_K(G)$. Results on matrix representations of S are derived.

[For further results see: Munn, Proc. Cambridge Philos. Soc. **51** (1955), 1–15; MR **16**, 561; Ponizovskiĭ, Mat. Sb. N.S. **38(80)** (1956), 241–260; MR **18**, 378; M. Teissier, C. R. Acad. Sci. Paris **234** (1952), 2413–2414, 2511–2513; MR **14**, 10; and Hewitt and Zuckerman, Acta Math. **93** (1955), 67–119; MR **17**, 1048.]

Št. Schwarz (Bratislava)

7074:

Post, K. A. Rank functions on semigroups. Nederl. Akad. Wetensch. Proc. Ser. A **61**=Indag. Math. **20** (1958), 332–334.

Let S be a semigroup and V the set of all infinite sequences $\{\alpha_1, \alpha_2, \alpha_3, \cdots\}$ of integers $\alpha_i \geqq 0$. V is considered as partially ordered in an obvious manner. A rank function is a mapping $\rho: S \to V$ such that 1. $\rho(ab) \leqq \rho(a)$, $\rho(ab) \leqq \rho(b)$, 2. $\rho(a) < \rho(b) \Rightarrow a = xby$ for some $x, y \in S$.

An equivalence relation in the set of all rank functions is introduced by the requirement $\rho_1 \sim \rho_2$ if and only if $\rho_1(a) \leqq \rho_1(b) \Leftrightarrow \rho_2(a) \leqq \rho_2(b)$. If the corresponding set of equivalence classes $\bar\rho$ is denoted by $\mathfrak{A}$ and $\mathfrak{A}$ is countable, then it contains a uniquely determined element $\bar\sigma$ such that for each $\bar\rho \in \mathfrak{A}$ we have $\bar\rho \preceq \bar\sigma$. (Here the ordering $\bar\rho_1 \preceq \bar\rho_2$ is defined by $\rho_2(a) \leqq \rho_2(b) \Rightarrow \rho_1(a) \leqq \rho_1(b)$ for some $\rho_1 \in \bar\rho_1$, $\rho_2 \in \bar\rho_2$.) The class $\bar\sigma$ is called the rank of S.

If S is a semigroup with identity element and we define $a \equiv b$ if both $a = xby$ and $b = taz$ hold for some $x, y, t, z \in S$, then $a \equiv b \Leftrightarrow \rho(a) = \rho(b)$ for $\rho \in \bar\sigma$.

The rank constructed constitutes an abstraction of the concept of rank as used for a full matrix ring over a commutative field.

Št. Schwarz (Bratislava)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 7228, 7229.

7075:

Allamigeon, André-Claude. Quelques propriétés des espaces homogènes réductifs à groupe nilpotent. C. R. Acad. Sci. Paris **247** (1958), 628–631.

Homogeneous spaces G/H are considered with a linear connexion. Among the results stated is the following: If G/H is symmetric then it is simply harmonic if and only if G is nilpotent. Now let G/H be reductive and G' a closed invariant subgroup whose Lie algebra is invariant under the projection of the Lie algebra of G onto the tangent space to G/H. Let $\tilde G/\tilde H = (G/G')/(HG'/G')$. Various relations between G/H and $\tilde G/\tilde H$ are considered including conditions under which G/H is an extension of $\tilde G/\tilde H$ in the sense of Patterson and Walker.

W. Ambrose (Cambridge, Mass.)

7076:

Ono, Takashi. Sur les groupes de Chevalley. J. Math. Soc. Japan **10** (1958), 307–313.

Chevalley [Tôhoku Math. J. (2) **7** (1955), 14–66; MR **17**, 457] has defined by Lie theory a group G_K over an arbitrary field, G_K being a subgroup of the group of automorphisms $A(\mathfrak{g}_K)$ of the Lie algebra $\mathfrak{g}_K$ over K obtained by tensorization with K of a semisimple Lie algebra $\mathfrak{g}$ over the complex field. The author first proves that the group G_K thus obtained is algebraic and connected and that, if L is an extension of K, G_L is obtained from G_K by extending to L the field of scalars. He then shows that for any prime number p, if Ω_p is the algebraic closure of the prime field of characteristic p, the group G_{Ω_p} is the connected component of the group $G^{(p)}$ obtained by reduction mod p from the group G_Q (Q being the rational field). Furthermore, $G^{(p)}$ is actually equal to G_{Ω_p} for almost all p. Finally, if either K has characteristic 0, or its characteristic $p>0$ does not divide the discriminant of the Killing form of $\mathfrak{g}$, the author proves that G_K is the connected component of $A(\mathfrak{g}_K)$ and that the Lie algebra of G_K is isomorphic to $\mathfrak{g}_K$. Actually, it is easy to see (using the theory of formal Lie groups of the reviewer) that the Lie algebra of G_K is isomorphic to $\mathfrak{g}_K$ if and only if $\mathfrak{g}_K$ has no center. *J. Dieudonné* (Paris)

TOPOLOGICAL ALGEBRA

See also 7060, 7070, 7214.

7077:

Numakura, Katsumi. Naturally totally ordered compact semigroups. Duke Math. J. **25** (1958), 639–645.

The author obtains a number of structure theorems for naturally totally ordered compact semigroups. He determines the structure completely in case the semigroup S has the property that $Se = eS$ for all idempotents e (this always holds if S is connected). His results extend work of A. H. Clifford [Amer. J. Math. **76** (1954), 631–646; MR **15**, 930]. *A. Shields* (Ann Arbor, Mich.)

7078:

Łoś, J.; and Schwarz, Š. Remarks on compact semigroups. Colloq. Math. **6** (1958), 265–270.

The authors begin with a special case of the following:

If $f: X \to Y$ is continuous, if $\mathscr{A}$ is a filter base of compact subsets in X and if $\mathscr{B}$ is a family of subsets of Y such that for each $B \in \mathscr{B}$ there is an $A \in \mathscr{A}$ with $B \supseteq f(A) \in \mathscr{B}$, then $f(\bigcap \mathscr{A}) = \bigcap \mathscr{B}$. This result is an immediate consequence of the well-known fact that $f(\bigcap \mathscr{A}) = \bigcap f(\mathscr{A})$. This is applied to prove some known results concerning compact topological semigroups. For example, the authors show that if S is a compact semigroup and if $a \in S$, then $\bigcap_n a^n S$ is invariant under left multiplication by a and is maximal relative to this property. This is a special case of a familiar but only recently published result of R. J. Koch [Proc. Amer. Math. Soc. **8** (1957), 397–401; MR **19**, 290]. A more general version of the maximal property is pointed out by A. D. Wallace [Indag. Math. **18** (1956), 271–274; MR **18**, 14]. In addition, the authors obtain new results on compact semigroups and the existence of kth roots, $k=1, 2, \cdots$, for elements of the semigroup. In particular, they show that, in a compact semigroup which has the property that for each x in S there is a kth root of x, $k=1, 2, \cdots$, each maximal subgroup enjoys the same property. *A. D. Wallace and A. Lester* (New Orleans, La.)

7079:

Šulka, Robert. On the maximal common refinement and the minimal common covering of two topological factoroids. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **8** (1958), 20–26. (Slovak. Russian and English summaries)

Conditions are given under which the maximal common refinement or, respectively, the minimal common covering of two topological factoroids on a topological groupoid is a topological factoroid [for notions involved cf. R. Šulka, same Časopis **5** (1955), 10–21; **6** (1956), 137–142; **7** (1957), 143–157; MR **16**, 997; **20**#2401, #5246].

M. Katětov (Prague)

FUNCTIONS OF REAL VARIABLES

See also 7234.

7080:

Adamson, Iain T. Transformations of integrals. Amer. Math. Monthly **65** (1958), 590–596.

The author presents another treatment of the change of variable theorem. The method is virtually the same as that to be found in the calculus books by Courant and by Ostrowski, except that more effort is made to use the notation of functional analysis. {This approach does not seem to be preferable to that given by J. Schwartz [same Monthly **61** (1954), 81–85; MR **15**, 611] or that presented by R. C. Buck in his book *Advanced calculus* [McGraw-Hill, New York-Toronto-London, 1956; MR **19**, 732]. These adopt a more intuitively satisfactory geometric framework, and in the latter, the proof is reduced to a simple and useful case of the Radon-Nikodym theorem; a treatment of oriented integrals, both single and multiple, is also to be found in the last named reference.}

K. de Leeuw (Stanford, Calif.)

7081:

Stamate, I. Une classe de formules de moyenne. Com. Acad. R. P. Romîne **8** (1958), 19–22. (Romanian. Russian and French summaries)

Several known mean-value theorems are given equivalent formulations in terms of tangents to parametric curves. *C. Davis* (Providence, R.I.)

7082:

Petrův, Vladimír. Über die symmetrische Ableitung stetiger Funktionen. Časopis Pěst. Mat. **83** (1958), 336–342. (Czech. Russian and German summaries)

The upper and lower symmetrical derivatives of $x(t)$ are defined by $x^s(t) = \lim\sup\{x(t+h)-x(t-h)\}/(2h)$, $x_s(t) = \lim\inf\{x(t+h)-x(t-h)\}/(2h)$ as $h \to 0$; the symmetrical derivative exists if $x^s(t)$, $x_s(t)$ are equal, the values $\pm\infty$ not being excluded. The main special case of the author's result is that the set of $x(t) \in C\langle 0, 1\rangle$ admitting a symmetrical derivative for at least one t, $0<t<1$, is of the first Baire category in $C\langle 0, 1\rangle$; there is in fact a residual set for which $x^s(t) = +\infty$, $x_s(t) = -\infty$, for all such t. More generally, the denominator $2h$ can be replaced by $\phi(h)$, subject to $h\phi(h) > 0$ for $h \neq 0$, $\phi(h) \to 0$ as $h \to +0$.

F. V. Atkinson (Canberra City)

7083:

Zaubek, Othmar. Über ein Verfahren in der Theorie der impliziten Funktionen und Extremwerte. Math. Ann. **137** (1959), 167–208; errata, 477.

Let f be a real single-valued function of two real variables x and y in a neighborhood U of $(0, 0)$. Assume that f is n-fold continuously differentiable in U, where n is not less than 2. Suppose that k is a natural number between 2 and n such that f together with all of its partial derivatives of order less than k vanish at $(0, 0)$ but f has at least one partial derivative of order k which does not vanish at $(0, 0)$. Define

$$g_0(z) = k^{-1} \sum_{j=0}^{k} \binom{k}{j} \left(\frac{\partial^k f}{\partial x^j \partial y^{k-j}} \right)_{(0,0)} z^{k-j}.$$

In terms of the zeros of g_0, methods are developed which enable one to determine in detail the nature of f at $(0, 0)$: whether f has a local extremum there and, if so, its type; or whether the equation $f(x, y) = 0$ possesses solutions for either x or y in a vicinity of $(0, 0)$ and, if so, the number of such solutions together with information about their continuity and differentiability properties. These methods are extended to analytic complex-valued functions of two complex variables. Ways to apply these methods to functions of more than two variables are indicated.

P. V. Reichelderfer (Columbus, Ohio)

MEASURE AND INTEGRATION

See also 7008, 7206, 7234, 7254.

7084:

Ellis, H. W. On the limits of Riemann sums. J. London Math. Soc. **34** (1959), 93–100.

Let $F(t)$ be a bounded function from $0 \leq t \leq 1$ to a normed vector space V. Let

$$\Sigma\,(D, E) = \sum_{i=1}^{n} F(\xi_i)(t_i - t_{i-1})$$

be a Riemann approximating sum, where D: $0 = t_0 < \cdots < t_n = 1$, E is a set $\xi_1, \cdots, \xi_n$ with $t_{i-1} \leq \xi_i < t_i$. Let $\Delta D = \max(t_{i+1} - t_i)$ and $R(F)$ be the set of points P which are limits of sequences of sums $\Sigma\,(D_n, E_n)$ with $\Delta D_n \to 0$. Using a simple measure-theoretic lemma, the author gives a proof of a result of the reviewer [Quart. J. Math. Oxford Ser. **18** (1947), 124–127; MR **9**, 137] that if V if finite-dimensional, then $R(F)$ is convex and that is $\varepsilon > 0$, there exists a $\delta > 0$ such that within an ε-distance of every point P of $R(F)$, there is a sum $\Sigma\,(D, E)$ with

$\Delta D<\delta$. [For extensions to inner product spaces, see Halperin and Miller, Trans. Roy. Soc. Canada, Sect. III (3) **48** (1954), 27–29; MR **16**, 596]. The author also considers some related (positive and negative) results in Banach spaces V.

P. Hartman (Baltimore, Md.)

7085:

Dionísio, J. J. On the uniqueness of measure extensions. Univ. Lisboa. Revista Fac. Ci. A (2) **6** (1957/58), 157–160.

Expository paper.

K. Krickeberg (Heidelberg)

7086:

Dionísio, J. J. On measure products. Univ. Lisboa. Revista Fac. Ci. A (2) **6** (1957/58), 305–310.

Expository paper; cf. N. G. de Bruijn and A. C. Zaanen, Nederl. Akad. Wetensch. Proc. Ser. A **57** (1954), 456–466 [MR **16**, 228].

K. Krickeberg (Heidelberg)

7087:

Newman, P. On a theorem of Urbanik. Fund. Math. **46** (1959), 231–234.

In this paper the following theorem is proved: If $\nu_1, \cdots, \nu_n$ are n non-atomic set functions defined on a certain σ-algebra M of subsets of a set E, satisfying the following conditions: (i) $\nu_i(X)\geqq 0$, all $X\in M$, $i=1, 2, \cdots, n$; (ii) $\nu_i(X)\leqq\nu_i(Y)$ whenever $X\subset Y$ and $\nu_i(X)<\nu_i(Y)$ if $\nu_i(Y-X)>0$, $i=1, 2, \cdots, n$; (iii) $\nu_i(X\cup Y)\leqq\nu_i(X)+\nu_i(Y)$, $i=1, 2, \cdots, n$, with strict inequality if either $\nu_i(X)$ or $\nu_i(Y)$ is non-zero; then there exists a partition $E=E_1\cup\cdots\cup E_n$ such that $\nu_i(E_i)\geqq\nu_i(E)/n$, with strict inequality for at least one i. If, moreover, the ν's satisfy also the following condition: For each $X\in M$ such that $\nu_i(X)=0$ for some i, we have $\nu_j(X)=0$, $j=1, 2, \cdots, n$, $i\neq j$; then there exists a partition for which the inequalities are all strict inequalities. If the measures are σ-additive, then this theorem follows from results of Urbanik [Fund. Math. **41** (1954), 150–162; MR **16**, 120].

It is questionable whether the theorem remains true if in (iii) the strict inequality condition is dropped but the following condition is added: There exists $X\in M$ and a pair of indices i, j such that $\nu_i(X)/\nu_i(E)\neq\nu_j(X)/\nu_j(E)$. The paper is motivated by the measure-theoretic formulation of the Steinhaus cake problem.

W. A. J. Luxemburg (Pasadena, Calif.)

7088:

Choksi, J. R. On compact contents. J. London Math. Soc. 33 (1958), 387–398.

Two theorems on the generation of measures by contents are proved. Theorem 1. Let $\mathscr{C}$ be a lattice of subsets of a set X, which is compact in the sense that $\bigcap_{n=1}^{\infty} C_n=\phi$ implies $\bigcap_{n=1}^{N} C_n=\phi$ for some N, and such that ϕ is in $\mathscr{C}$. Let λ be a "regular content" on $\mathscr{C}$, i.e., a positive real-valued function on $\mathscr{C}$ which is monotone, subadditive, additive, and such that if $C\subseteq D$ and $\varepsilon>0$, there is a C' in $\mathscr{C}$ with $C'\subseteq D-C$ such that $\lambda(C)+\lambda(C')>\lambda(D)-\varepsilon$. Then λ can be extended to a measure m on the σ-ring $S(\mathscr{C})$ generated by $\mathscr{C}$, m being compact in the sense that if $\varepsilon>0$ and M is in $S(\mathscr{C})$ with $m(M)<\infty$, there exists a C in $\mathscr{C}_\delta$ such that $C\subseteq E$ and $m(E)-m(C)<\varepsilon$. In the second theorem the author applies theorem 1 to obtain regular measures on locally compact spaces which are not necessarily Hausdorff.

E. Nelson (Princeton, N.J.)

7089:

Eggleston, H. G. On measureless sets. Proc. London Math. Soc. (3) **8** (1958), 631–640.

A subset X of n-space R^n is called measureless if no non-atomic metric outer measure on its subsets can assume a positive finite value. X is called s-dimensional if its t-dimensional Hausdorff measure $\Lambda_t(X)$ is zero for $t>s$ and infinite for $t<s$. The principal result is that if $\aleph_1=2^{\aleph_0}$, then every set X with $0<\Lambda_s(X)<\infty$ contains a measureless s-dimensional subset. It is first shown that subsets, countable unions, and cartesian products of measureless sets are measureless. With the aid of the continuum hypothesis and suitable mappings in R^n a measureless 1-dimensional linear set is then constructed and mapped onto an appropriate subset of X.

J. C. Oxtoby (Bryn Mawr, Pa.)

7090:

Tsurumi, Shigeru. On the ergodic theorems concerning Markov processes. Tôhoku Math. J. (2) **10** (1958), 146–164.

Let $P(x, A)$ be the transition function of a Markov process on an abstract measurable space X, S the operator on measures given by $S\phi(A)=\int P(x, A)\phi(dx)$ and T the operator on functions given by $Tf(x)=\int f(y)P(x, dy)$. Let μ be a finite measure such that $\mu(A)=0$ implies $S\mu(A)=0$. The author establishes necessary and sufficient conditions for various formulations of the individual and mean ergodic theorems to hold on the spaces $L_p(\mu)$, $p\geqq 1$. For the case of an invariant measure μ, i.e., $S\mu=\mu$, this was done by E. Hopf [J. Math. Mech. 3 (1954), 13–45; MR **15**, 636] and others. The weakest of the author's three conditions, denoted (C.1), is that there exists a positive constant K such that $\alpha(A)=\limsup_{n\to\infty} n^{-1}\sum_{j=0}^{n-1} S^j\mu(A)\leqq K\mu(A)$ for all A. This implies an individual ergodic theorem extending Hopf's theorem. The method of proof is to reduce the problem to Hopf's theorem by proving the existence of a finite invariant measure λ closely related to μ. In fact, if (C.1) holds, there exists a finite invariant measure λ such that $\alpha(A)\leqq\lambda(A)\leqq K^2\mu(A)$ for all A. The author calls a set A compressible (μ) in case $P(x, A)\leqq e_A(x)$ for μ-a.a. x and $P(x, A)<e_A(x)$ for x in a set of positive μ-measure, where e_A is the indicator (characteristic function) of A. A set A is called incompressible (μ) if $P(x, A)\leqq e_A(x)$ for μ-a.a. x and if A contains no compressible (μ) set. He proves the decomposition theorem: X splits into two disjoint sets Y and Z such that Y is incompressible (μ) and Z contains no incompressible (μ) set of positive measure.

E. Nelson (Princeton, N.J.)

7091:

Varadarajan, V. S. On a theorem of F. Riesz concerning the form of linear functionals. Fund. Math. **46** (1959), 209–220.

This paper provides a very neat approach to the F. Riesz representation theorem for linear functionals on spaces of continuous functions, in which the classical integration theory is reduced to a minimum. X denotes a topological space, $C(X)$ the space of real, bounded, continuous functions on X, $L(X)$ the subspace of functions with compact supports.

A compact Hausdorff X is said to have property R if every positive linear functional (PLF) Λ on $C(X)=L(X)$ is representable as an integral with respect to some Baire measure on X. It is shown that property R is stable under the following processes: (1) passage to a continuous image; (2) passage to a closed subspace; (3) passage to a product in which every finite partial product has property

R. The Riesz theorem for every compact Hausdorff X then follows in view of the fact that such an X is the continuous image of a closed subspace of a product of two-point discrete spaces.

The locally compact Hausdorff case is reduced to the above by the following device. If Λ is a PLF on $L(X)$ and $K\subset X$ is compact, for each positive continuous f on K define $\Lambda_K(f)=\lim_{n\to\infty}\Lambda(f_n)$, where $(f_n)\subset L(X)$ and $f_n\downarrow f$ on K, $f_n\downarrow 0$ on $X-K$. The "compact case" is applied to each Λ_K, giving representative Baire measures m_K on K; the various m_K are compatible and combine to give a Baire measure m on X representing Λ.

A general X is then considered. It is shown that local compactness and separation properties of X play the role merely of ensuring that $L(X)$ is sufficiently wide. In fact, suppose that L is an ideal in $C(X)$ (i.e. L is linear; and $f\in L$, $g\in C(X)$, $|g|\leq|f|$ imply $g\in L$). We say that a PLF Λ on L is an integral if there exists a measure m on the minimal σ-ring $S(L)$ with respect to which each function in L is measurable such that $\Lambda(f)=\int f dm$ for f in L. Further, Λ is said to be σ-smooth if $f_n\in L$ and $f_n\downarrow 0$ imply $\Lambda(f_n)\downarrow 0$. It is then shown that any PLF Λ on L is an integral in either of the following cases: X compact (not necessarily Hausdorff) and Λ is σ-smooth on L; X general and $L=L(X)$.

The author raises the question: If X is compact Hausdorff and L is an ideal in $C(X)$, when is it true that all PLFs on L are σ-smooth? He shows that a sufficient condition is that $f\in L$ shall entail $f^{\frac{1}{2}}\in L$.

R. E. Edwards (London)

FUNCTIONS OF A COMPLEX VARIABLE

See also 7154, 7162, 7164, 7165, 7181, 7197, 7198, 7215.

7092:

Gleason, Andrew M. A metric for the space of function elements. Amer. Math. Monthly **65** (1958), 756–758.

Let e denote a function element $\sum a_n(\zeta-z)^n$, centered at z, and let $R(e)=\liminf|a_n|^{1/n}$. Then $R(e)$ is the radius of convergence of e, and $\{\zeta:|\zeta-z|<R(e)\}$ is the circle of convergence of e. Denote by $p(e)$ the center z of e. Two elements are called adjacent if their circles of convergence overlap and the functions defined by them coincide in the common region of convergence. The author defines $\rho(e_1, e_2)$ to be $R(e_1)+R(e_2)$ if e_1 and e_2 are not adjacent and to be $|p(e_1)-p(e_2)|$ if they are adjacent. He then shows that ρ is a (possibly infinite-valued) metric for the space E of function elements, and that the topology induced by ρ is the usual topology for E given by analytic continuation. The metric ρ has the advantage over the usual metric defined in terms of analytic continuation, in that $\rho(e_1, e_2)$ is immediately determined from e_1 and e_2 and does not depend on a possible analytic continuation from e_1 to e_2.

H. L. Royden (Zürich)

7093:

Wilson, R. The Hadamard product of an integral function and a power series. J. London Math. Soc. **33** (1958), 398–403.

Denote by $F_1\odot F_2$ the Hadamard product of $F_1(z)$ and $F_2(z)$. In a previous paper [same J. **32** (1957), 421–429; MR **19**, 949] the author has studied the case in which F_1 and F_2 are entire functions, and the connection between the directions of strongest growth of F_1 and F_2, and those of $F_1\odot F_2$. The same techniques may be applied to study the case in which F_1 is entire and F_2 is a suitable power series; the comparison is now between directions of strongest growth for F_1 and singular directions for F_2, and the results are quite analogous to those in the paper cited.

R. C. Buck (Stanford, Calif.)

7094:

Wilson, R. On Hadamard composition with algebraic-logarithmic singularities. Quart. J. Math. Oxford Ser. (2) **9** (1958), 68–72.

Let $f(z)=\sum a_nz^n$, $g(z)=\sum b_nz^n$, $h(z)=\sum a_nb_nz^n$ in $|z|<1$. The author proves that if the sole singularity of f in the finite plane is an algebraic-logarithmic point at α with a single dominant element, and if β is a singularity of g on $|z|=1$, then the point $\alpha\beta$ is singular for h. Under certain conditions, the theorem can be extended to the case where α and β are vertices of the principal star-domains of f and g.

G. Piranian (Ann Arbor, Mich.)

7095:

Pogorzelski, W. Problème généralisé de Hilbert pour les arcs non fermés. Ann. Sci. École Norm. Sup. (3) **75** (1958), 201–222; errata, 409.

In the complex plane let L denote a set of p directed arcs a_ib_i, $i=1, 2, \cdots, p$, not closed and having no points in common, but having continuous tangents. The author solves the problem of finding n functions $\phi_1(z), \phi_2(z), \cdots, \phi_n(z)$, holomorphic in the domain exterior to the cuts formed by the arcs, whose limit values $\phi_\nu^+(t)$ and $\phi_\nu^-(t)$ from the two sides of the cuts satisfy the system of n given non-linear relations

$$\phi_\nu^+(t)=G_\nu(t)\phi_\nu^-(t) +\lambda F_\nu[t, \phi_1^+(t), \cdots, \phi_n^+(t), \phi_1^-(t), \cdots, \phi_n^-(t)],$$

$\nu=1, 2, \cdots, n$, at each point t of the arcs of L except for the extremities. The given functions G_ν and F_ν satisfy certain Hölder conditions and the modulus of the parameter λ must be chosen sufficiently small. The functions $\phi_\nu(z)$ are determined by means of a system of non-linear integral equations. The existence of the solution is shown by an application of a fixed point theorem of J. Schauder.

M. S. Robertson (New Brunswick, N.J.)

7096:

Pogorzelski, W. Sur certaines classes de fonctions complexes définies sur les arcs non fermés. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 57–62.

The notation of the previous review will be used here. Let $\mathfrak{H}_\alpha^\mu$ denote the set of all functions $\phi(t)$ of the complex variable t continuous at each interior point of the set of arcs comprising L and which verify the inequalities

$$|\phi(t)|<A\left[\prod_{\nu=1}^{p}|t-a_\nu|\cdot|t-b_\nu|\right]^{-\alpha},$$

$$|\phi(t)-\phi(t_1)|<B|t-t_1|^\mu\left[\prod_{\nu=1}^{p}|t-a_\nu|\cdot|t_1-b_\nu|\right]^{-\alpha-\mu},$$

$0<\mu<1$, $0\leq\alpha<1$, $\alpha+\mu<1$, A and B positive constants. Let $\mathfrak{H}_\alpha$ denote the union of the classes $\mathfrak{H}_\alpha^\mu$, α fixed, μ varying over $(0, 1)$ and let $\mathfrak{H}^\mu$ be the union of the classes $\mathfrak{H}_\alpha^\mu$ for μ fixed and α varying over $[0, 1)$. $\mathfrak{H}$ denotes the union of all the classes $\mathfrak{H}_\alpha^\mu$ as α and μ both vary.

The author studies the properties of these and other analogously defined classes and makes applications to the problem of inversion for the singular Cauchy transfor-

mation

$$f(t)=\int_L \frac{\phi(\tau)}{\tau-t}\,d\tau,$$

obtaining explicit formulae for all the solutions $\phi(t)$ of his well-defined classes $\mathfrak{H}(c_{k_1}, \cdots, c_{k_s})$.

M. S. Robertson (New Brunswick, N.J.)

7097:

Przeworska-Rolewicz, D. Problème non linéaire d'Hilbert pour un système infini de fonctions. Ann. Polon. Math. **5** (1958/59), 293–301.

In a plane of one complex variable let $L_1, L_2, \cdots, L_p$ denote p Jordan curves, non-intersecting and bounding disjoint domains $S_1^-, S_2^-, \cdots, S_p^-$. Let a Jordan curve L_0 surround the curves $L_1, \cdots, L_p$ and have no point in common with them. All the curves $L_0, \cdots, L_p$ have continuous tangents. S_0^- denotes the exterior of L_0, and S^+ the domain bounded by L_0 and the curves $L_1, \cdots, L_p$.

The author solves the following problem. To find an infinite sequence of functions $\{\phi_n(z)\}$ each holomorphic in the domains $S^+, S_0^-, \cdots, S_p^-$ separately, having limit values ϕ_n^+, ϕ_n^- relative to these domains, and also satisfying at each point t of the set $L=L_0+L_1+\cdots+L_p$ the relations

$$\phi_n^+(t)=G_n(t)\phi_n^-(t)+F_n[t, \phi_1^+(t), \phi_1^-(t), \phi_2^+(t), \phi_2^-(t), \cdots]$$

for $n=1, 2, \cdots$. The problem is solved under the hypothesis that the functions $G_n(t)$, $F_n(t, u_1, u_2, \cdots)$ satisfy the conditions:

$$|G_n(t)-G_n(t')|\leqq g_n|t-t'|^\mu$$
$$(0<\mu<1,\ n=1, 2, \cdots, t\in L,\ G_n(t)\neq 0);$$

for $t\in L$, $|u_n|\leqq R$,

$$|F_n(t, u_1, \cdots)-F_n(t', u_1', \cdots)|\leqq f_n[|t-t'|^\mu+\sum_{i=1}^\infty \alpha_{ni}|u_i-u_i'|]$$
$$(0<\mu<1, n=1,2,\cdots, \alpha_{ni}>0, \sum_{i=1}^\infty \alpha_{ni}<1).$$

The author had previously solved an analogous problem for a finite system of functions [same Ann. **2** (1955), 1–13, 136–144; MR **19**, 24; **18**, 728]. The problem is solved by application of a fixed point theorem to a system of non-linear integral equations of the second kind for suitable choice of constants involved. The formulae are too involved to be given here.

M. S. Robertson (New Brunswick, N.J.)

7098:

Walsh, J. L. On infrapolynomials with prescribed constant term. J. Math. Pures Appl. (9) **37** (1958), 295–316.

This paper is concerned mainly with a polynomial

$$P_n(z)=z^n+A_1z^{n-1}+\cdots+A_{n-1}z+A_n$$

(A_n prescribed), which is an infrapolynomial on a given compact point set E of at least n points, in the sense that no polynomial

$$q_n(z)=z^n+B_1z^{n-1}+\cdots+B_{n-1}z+A_n$$

is an underpolynomial of $P_n(z)$ on E. If $P_n(z)\neq 0$ on E, geometric considerations lead to the existence of n points z_k in E and m positive numbers λ_k (with $n-1\leqq m\leqq 2n-1$) such that

$$\sum_{k=1}^m \lambda_k[z_k^s/p_n(z_k)]=0\ \ (s=1, 2, \cdots, n-1).$$

From this may be inferred that the sum of the angles subtended by E in the zeros of $P_n(z)$ is not less than π. With the aid of theorems on the location of the zeros of the derivative of a polynomial, the additional theorems proved include the following. (1) Let E lie in the half-plane $x>0$ and let, corresponding to each non-real point z_k of E, a circle C_k be drawn tangent at z_k and $\bar{z}_k$ to the lines joining these points to the origin; then no non-real zero of $P_n(z)$ lies exterior to all the C_k. (2) If E lies on the circle $|z|=R$ and if $|A_n|=R^n$, then all zeros of $P_n(z)$ also lie on this circle.

M. Marden (Milwaukee, Wis.)

7099:

Zervos, Spiros. Généralisation de la notion de "domaine circulaire" du plan complexe; applications. C. R. Acad. Sci. Paris **246** (1958), 2706–2709.

The author denotes by K a field (not necessarily commutative), by K_ω the union of K with the infinite element and by $\phi_\zeta(A)$ the mapping of a subset A of K_ω via the transformation $\phi_\zeta(z)=(\zeta-z)^{-1}$. His generalization of a circular domain is a subset A of K_ω which, if not empty and not K or K_ω, is such that, for every $\zeta\in K-A$, $\phi_\zeta(A)$ is closed relative to addition and both left and right multiplication. For these generalized circular domains, the author states generalizations of a number of theorems from classical geometry of the zeros of polynomials, including the Gauss-Lucas, Laguerre-Walsh and Grace theorems.

M. Marden (Milwaukee, Wis.)

7100:

Oğuztöreli, M. Namik. Un exemple de fonctions de la classe de Mr. Elfving. Rev. Fac. Sci. Univ. Istanbul Sér. A **22** (1957), 13–24. (Turkish summary)

Denote by $\{w, z\}$ the Schwarzian derivative of the function $w=w(z)$. The present paper considers mappings defined by the solutions of $\{w, z\}=2[Az^{-2}+Bz^{n-2}+Cz^{2n-2}]$, with $n\geqq 3$, $A=(1-k^2)/4$, k an integer. The logarithmic and algebraic ramification points of the function inverse to w are determined, and in certain special cases the solutions are expressed in terms of generalized Laguerre polynomials. The Riemann surface of these functions gives a concrete example of a general class considered by Elfving [Acta Soc. Sci. Fennicae. N.S. **2** (1934), no. 3, 1–60].

H. L. Royden (Zürich)

7101:

Ozawa, Mitsuru. On extremal quasiconformal mappings. Kōdai Math. Sem. Rep. **10** (1958), 109–112.

Let W and W' be two closed Riemann surfaces of the same genus $g\geqq 2$, and let p_0 be a fixed point on W. Let $K(q)$ be the minimum in a given homotopy class of the maximum dilation of the quasi-conformal mappings of W onto W' which take p_0 into q. Then $K(q)$ is a continuous function on W', and the author applies Morse theory to obtain information about the critical set of $K(q)$. In particular, he obtains the result that there is only one point at which $K(q)$ has a relative minimum.

H. L. Royden (Zürich)

7102:

Royden, H. L. Open Riemann surfaces. Ann. Acad. Sci. Fenn. Ser. A. I, no. 249/5 (1958), 13 pp.

This is an address at the Colloquium on the Theory of Functions, Helsinki, 1957. The author surveys three aspects of the theory of open Riemann surfaces as follows. (1) Classification of open Riemann surfaces: inclusion relations of O-classes, elliptic differential equations of the type $\Delta u-Pu=0$ on Riemann surfaces, invariance of O-classes under quasi-conformal mappings,

surfaces with identical ideal boundaries. An example is given to show that the classes O_{AD} and O_{AB} are not preserved under quasi-conformal mappings. (2) The structure of function algebras on Riemann surfaces: characterization of the conformal type by the field of meromorphic functions, the ring of analytic functions, the ring of bounded analytic functions. (3) Compactification of open Riemann surfaces: Kerékjártó-Stoilow and Martin compactifications, Brownian motion on Riemann surfaces, prime ends on Riemann surfaces, use of Gelfand's theory of normed rings. A compactification is introduced which is preserved under quasi-conformal mappings and is useful for the study of HD-functions.

L. Sario (Los Angeles, Calif.)

7103:

Raleigh, John. The Fourier coefficients of the invariants $j(2^{\frac{1}{2}};\tau)$ and $j(3^{\frac{1}{2}};\tau)$. Trans. Amer. Math. Soc. **87** (1958), 90–107.

Let $G(\lambda_q)$ denote the properly discontinuous group generated by the two substitutions $S(\tau)=\tau+\lambda_q$ and $T(\tau)=-1/\tau$, where $\lambda_q=2\cos(\pi/q)$ and q is an integer $\geqq 3$. The class of groups $\{G(\lambda_q)\}$ was initially studied by E. Hecke [Math. Ann. **112** (1936), 664–699]. $G(\lambda_3)$ is the full modular group $G(1)$, the fundamental invariant of which, $j(1;\tau)=12^3J(1;\tau)$, has been widely studied. An invariant for $G(\lambda_4)$, here denoted by $j(2^{\frac{1}{2}};\tau)$, was obtained by J. W. Young [Trans. Amer. Math. Soc. **5** (1904), 81–104] as a quotient of theta-null series, and an invariant $j(3^{\frac{1}{2}};\tau)$ for $G(\lambda_6)$ was obtained by J. I. Hutchinson [Trans. Amer. Math. Soc. **3** (1902), 1–11], also as a quotient of theta-null series. In the present paper the author obtains convergent series for the Fourier coefficients of the invariants $j(2^{\frac{1}{2}};\tau)$ and $j(3^{\frac{1}{2}};\tau)$. The method is an extension of that used by H. Rademacher [Amer. J. Math. **60** (1938), 501–512] for obtaining the Fourier coefficients of the invariant $j(1;\tau)$.

W. H. Simons (Vancouver, B.C.)

7104:

Petersson, Hans. Über Betragmittelwerte und die Fourier-Koeffizienten der ganzen automorphen Formen. Arch. Math. 9 (1958), 176–182.

Let $f(\tau)$, an automorphic form of negative dimension $-r$ on a horocyclic group Γ, have the expansion $f(\tau)=\sum_{n=0}^{\infty} b_{n+\kappa}\exp(2\pi i(n+\kappa)\tau/N)$ at $\tau=i\infty$, where κ, N are constants depending on r, Γ, and the multiplier system v. In a short and simple discussion the author establishes the following estimates for the Fourier coefficients $b_{n+\kappa}$. (1) $b_{n+\kappa}=O(n^{r-1})$, for $r>2$; (2) $b_{n+\kappa}=O(n\log n)$, for $r=2$; $b_{n+\kappa}=O(n^{r/2})$, for $0<r<2$, $r\neq 2^{-h}$ $(h=0, 1, 2, \cdots)$; $b_{n+\kappa}=O(n^{r/2}\log^{r/2} n)$, $r=2^{-h}$ $(h=0, 1, 2, \cdots)$. It can be shown by suitable examples involving Eisenstein series that (1) and (2) cannot be essentially improved.

The principal tool used in the author's elegant proof is the estimation of the partial sums of the Poincaré series in the cases in which the series diverges. The author accomplishes this by applying simple concepts of hyperbolic geometry, just as Poincaré did in proving convergence for $r>2$.

{Reviewer's comment: For automorphic forms which vanish at $\tau=i\infty$, these estimates can be sharpened. The proof, which was given by Hecke in the case of the modular group and its congruence subgroups, is applicable to the general herocyclic group Γ and runs as follows. Let $\tau=x+iy$. Note that $\varphi(\tau)=|y^{r/2}f(\tau)|$ is invariant under Γ. Thus $\varphi(\tau)$ assumes all its values in a fundamental region of Γ and so is bounded in the upper half-plane, since $f(i\infty)=0$ to exponential order (i.e., either $\kappa>0$, or $\kappa=0$ and $b_0=0$). This shows that $f(\tau)=O(y^{-r/2})$ as $y\to 0$, uniformly in x. Now $a_n=(2\pi i)^{-1}\int_L f(\tau)\exp(-2\pi i(n+\kappa)\tau/N)\cdot dx$, where L is the segment $0\leqq x<\kappa$, $y=y_0>0$. Hence, $a_n=O(y_0^{-r/2}$ $\exp(2\pi(n+\kappa)y_0/N)$, and for $y_0=n^{-1}$, this becomes (3) $a_n=O(n^{r/2})$, $r>0$. For certain groups, this bound can be further improved. Rankin showed in 1939 [Proc. Cambridge Philos. Soc. **35** (1939), 351–372; MR **1**, 69] that if $f(\tau)$ is a modular form (belonging to the full group or a congruence subgroup) which vanishes at infinity, then $a_n=O(n^{1/2-1/5})$, provided $\kappa=0$. By employing A. Weil's estimates for the Kloosterman sums, this can be reduced to $a_n=O(n^{r/2-\frac{1}{4}+\epsilon})$. (In the case of the modular group, $\kappa=0$ forces r to be an even integer.)}

J. Lehner (East Lansing, Mich.)

7105:

Tôki, Yukinari. Proof of Ahlfors principal covering theorem. Rev. Math. Pures Appl. **2** (1957), 277–280.

Admettant les deux premiers lemmes d'Ahlfors de sa théorie des recouvrements ($|S-S(D)|\leqq hL$ et $|S-S(\beta)|\leqq hL$) l'auteur déduit le théorème fondamental: $\rho^+\geqq\rho_0 S-hL$ grâce à un ingénieux découpage de la surface de recouvrement, et en se bornant au cas où la surface de base est de genre zéro.

Toutefois quelques explications seraient nécessaires pour justifier complètement la répartition des frontières suivant les morceaux découpés sur la surface de recouvrement.

L. Fourès (Marseille)

7106:

Fuchs, W. H. J. A theorem on the Nevanlinna deficiencies of meromorphic functions of finite order. Ann. of Math. (2) **68** (1958), 203–209.

The author proves the following theorem. Let $f(z)$ be a meromorphic function of finite lower order λ, and set $q=\max(2, \lambda)$. Then if δ_k are the defects in the sense of Nevanlinna, we have

$$\sum(\delta_k)^{\frac{1}{3}}\leqq(Aq\log q)^{\frac{1}{3}}.$$

No other inequality of such generality is known, apart from the Nevanlinna defect relation $\sum\delta_k\leqq 2$. The author states that an example of A. A. Goldberg [Dokl. Akad. Nauk. SSSR **98** (1954), 893–895; MR **17**, 144] can be modified to show that the index $\frac{1}{3}$ cannot be replaced by any smaller number. Also $q\log q$ cannot be replaced by anything smaller than q.

The proof is surprisingly simple and depends on an estimation of the variation of $\log|f'(z)|$ on the circle $|z|=r$.

W. K. Hayman (London)

7107:

af Hällström, Gunnar. Übertragung eines Satzkomplexes von Weierstrass und Dinghas auf beliebige Randmengen der Kapazität Null. Ann. Acad. Sci. Fenn. Ser. A. I, no. **250/12** (1958), 9 pp.

Some theorems by Dinghas [Math. Z. **66** (1957), 389–408; MR **19**, 539], sharpening the Casorati-Weierstrass theorem, are generalized for arbitrary sets of essential singularities with capacity zero. $[a, b]$ being the chordal distance between a and b on the Riemann sphere and $w(z)$ meromorphic in $r_0\leqq|z|<\infty$, denote $\mu(r, a)=\mathrm{Min}_{|z|=r}[w(z), a]$. Dinghas proved: (I) For every a and suitable arbitrarily large r, $\mu(r, a)<\exp(-\alpha T(r))$, with arbitrary $\alpha<1$ ($T(r)$ being the "local" Nevanlinna's characteristic function of $w(z)$). (II) If $\limsup_{r\to\infty}(h(r, a)/A(r))<1$, then (*) $\int_{r_0}^{\infty}\mu(r, a)r^{-1}dr$ converges. Here $n(r, a)$ is the number of a-points in the ring $r_0<|z|\leqq r$ and $A(r)$ the mean sheet-number of the image of this ring on the

Riemann sphere. The author generalizes both theorems for the above set of capacity zero. Theorem (II) becomes even sharper as the power 2 in (*) is replaced by any positive β. The method used is a modification of Dinghas' proof. An attempt to generalize this theorem by replacing the Nevanlinna-defect by the Valiron-defect fails, as a suitable example constructed by the author in a previous paper [Acta Acad. Abo. **12** (1940), no. 8; MR **2**, 275] shows. The converse of this theorem is not true, as another example (of Nevanlinna and Ullrich) shows.

B. A. Amirà (Jerusalem)

7108:

Schubart, Hans; und Wittich, Hans. Zur Wachstumsordnung der Lösungen einer Klasse nichtlinearer Differentialgleichungen. Arch. Math. **9** (1958), 355–359.

A differential equation $w''=P(z, w)$, where P is a polynomial in z and $w=w(z)$, has single-valued analytic solutions only if $P=6w^2+A_0+A_1z$ or $P=2w^3+(B_0+B_1z)w+C$. In these cases, the authors have previously studied the value distribution properties of the solutions [see Wittich, *Neuere Untersuchungen über eindeutige analytische Funktionen*, Springer, Berlin, 1955; MR **17**, 1067]. By virtue of certain results of Boutroux, they were able to show that the solutions are of finite order. In this paper, a more direct proof is given to this fact: If $\lambda=\limsup \log T(r, w)/\log r$, then for the solutions, $\lambda\leqq 3$.

O. Lehto (Helsinki)

7109:

Weston, J. D. Some theorems on cluster sets. J. London Math. Soc. **33** (1958), 435–441.

The author formulates and proves abstract versions of a theorem of Collingwood. The latter theorem, in a weak form, asserts that if f is continuous in the unit disk, then the radial cluster set of f at $e^{i\theta}$ coincides with the complete cluster set at $e^{i\theta}$, except at a set of points $e^{i\theta}$ of first category.

G. Piranian (Ann Arbor, Mich.)

7110:

Chen, Han-lin. Some theorems on typical real-functions. Progress in Math. **3** (1957), 452–461. (Chinese)

7111:

Gelfond, A. Sur une méthode générale pour les problèmes d'interpolation. Ann. Acad. Sci. Fenn. Ser. A. I. no. 251/4 (1958), 14 pp.

Let $\varphi_n(z)$ be regular outside the disc $|z-\alpha_n|\leqq p_n$. Define a linear functional L_n on the appropriate class of analytic functions f by

$$L_n(f)=\frac{1}{2\pi i}\int_{\Gamma_n}\varphi_n(z)f(z)dz$$

when Γ_n encloses D_n. Construct a sequence of polynomials P_n orthogonal to the $\{L_n\}$ in the sense that $L_k(P_n)=\delta_k{}^n$. Then, the author wishes to study the uniqueness and representation problems: when does $L_n(f)=0$ for all $n=0, 1, 2, \cdots$ imply $f=0$, and for which f is it true that $f(z)=\sum L_n(f)P_n(z)$? Suppose that the functions φ_n obey the following growth restriction:

$$\varphi_n(z)\leqq\frac{\omega_n}{r^{n-m_n+1}(r-r_n)^{m_n}}$$

for $|z|=r>r_n$, where $1\leqq\omega_0\leqq\omega_1\leqq\cdots$, $\lim m_n=\infty$, $r_n=O(1)$, and $\limsup \omega_n r_n m_n=\alpha_0<\infty$.

Let μ be the minimal root of the equation

$$1=x+\omega\sum_{2}^{\infty}k^{-k}\left(\frac{x}{\omega}\right)^k,$$

where $\omega=\liminf \omega_n$. If R obeys $R>e\alpha_0/\mu$, $R>R'\geqq r_n$, $n=0, 1, 2, \cdots$, then any function $f(z)$ which is analytic in $|z|<R$ has the convergent development $\sum L_n(f)P_n(z)$. For uniqueness, this leads to the condition $\lim \omega_n r_n m_n=0$, which improves the criterion Goncharov gave for the particular case $\varphi_n(z)=(z-z_n)^{-n-1}$. [See Ann. Sci. Ecole. Norm Sup (3) **47** (1930), 1–78.]

An analogous result is given for the development of entire functions into series. [Reference should also be made to J. M. Whittaker, *Sur les séries de base de polynomes quelconques*, Gauthier-Villars, Paris, 1949; MR **11**, 344; and to R. C. Buck, *Lectures on functions of a complex variable*, pp. 409–419, Univ. of Michigan Press, Ann Arbor, 1955; Trans. Amer. Math. Soc. **64** (1948), 283–298; Proc. Amer. Math. Soc. **6** (1955), 793–796; MR **17**, 140; **10**, 693; **17**, 356.]

R. C. Buck (Stanford, Calif.)

FUNCTIONS OF SEVERAL COMPLEX VARIABLES, COMPLEX MANIFOLDS

See 7017, 7083.

SPECIAL FUNCTIONS

See also 7190, 7523.

7112:

González, Mario O. Theory of elliptic functions. IV. Jacobi elliptic functions. Rev. Soc. Cubana Ci. Fís. Mat. **4** (1957/58), 57–63. (Spanish)

Continuing the (mainly didactical) presentation of elliptic functions [see same Rev. **3** (1953), 39–44; **3** (1954), 67–75; **3** (1955), 109–118; **3** (1956), 149–157; **4** (1957), 3–32; MR **15**, 421; **17**, 481; **19**, 739], the author expresses the Jacobi elliptic functions as rational functions of $\mathrm{Tan}(z/2)$, $\mathrm{Tan}\, z$ being the elliptic function defined in part III. One has $\mathrm{sn}\, z=2\,\mathrm{Tan}(z/2)\{1+\mathrm{Tan}^2(z/2)\}^{-1}$, whence the other definitions follow. The periods, poles, zeros, addition theorems, series expansions of sn z, cn z, dn z now easily follow from the corresponding ones of Tan z. A brief presentation of the ϑ-functions leads to the nice formula $\mathrm{Tan}\, z=\vartheta_1(v)\vartheta_3(v)/\vartheta_0(v)\vartheta_2(v)$, where $v=z/2K$. The chapter ends with remarks concerning the connection of Tan z with the exponential function, the proof of $\mathrm{Tan}\, z=\mathrm{tg}(\frac{1}{2}\,\mathrm{am}\, 2z)$ and a graphic interpretation.

E. Grosswald (Philadelphia, Pa.)

7113:

González, Mario O. Theory of elliptic functions. V. Weierstrass elliptic functions. Rev. Soc. Cubana Ci. Fís. Mat. **4** (1957/58), 77–86. (Spanish)

The Weierstrass elliptic functions are defined by setting $\wp(z)=e_1+\gamma^2/\mathrm{Tan}^2\,\gamma z$, where $\gamma^4=3e_1{}^2-\frac{1}{4}g_2$ and e_1 is one of the roots of the irreducible cubic $4t^3-g_2t-g_3=0$. Some of the theory easily follows from the properties of Tan z [see same Rev. **4** (1957), 3–32; MR **19**, 739]. However, the properties of the ζ and σ-functions are obtained, as usual, from $d^2(\log\sigma(z))/dz^2=d\zeta(z)/dz=-\wp(z)$. This expository chapter also contains the representation of elliptic functions as rational functions of the σ-function, or the $\wp$ and $\wp'$ functions. The series for $\zeta(z)$ and $\wp(z)$ and the Weierstrass product decomposition of $\sigma(z)$, addition theorems, etc., are also included. The chapter ends with some further relations between Tan z, $\sigma(z)$, $\wp(z)$ and $\wp'(z)$.

E. Grosswald (Philadelphia, Pa.)

7114:

Koschmieder, Lothar. Zweireihige Determinanten aus Thetafunktionen. Arch. Math. 9 (1958), 183–185.

Let $0<q<1$ and

$$\vartheta_1(v)=2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin(2n+1)\pi v,$$

$$\vartheta_2(v)=2\sum_{n=0}^{\infty}q^{(n+\frac{1}{2})^2}\cos(2n+1)\pi v$$

be theta-functions of the first two types. The author shows that the two determinants

$$D_1=\begin{vmatrix}\vartheta_1(v-w) & \vartheta_1(v)\\ \vartheta_1(v) & \vartheta_1(v+w)\end{vmatrix},\quad D_2=\begin{vmatrix}\vartheta_2(v-w) & \vartheta_2(v)\\ \vartheta_2(v) & \vartheta_2(v+w)\end{vmatrix}$$

are negative for all real values of v and w, except for integral values of w, in which case both determinants vanish. Similar determinants formed with the Jacobian elliptic functions sn u, cn u, and dn u are also studied.

W. Seidel (Notre Dame, Ind.)

7115:

Srivastava, Krishna Ji. Certain integral representation of Macrobert's E-function. Gaṇita 8 (1957), 51–60.

Pour la fonction

$$E\begin{pmatrix}a_1, & a_2, & \cdots, & a_p\\ & & & & ;\ x\\ \rho_1, & \rho_2, & \cdots, & \rho_q\end{pmatrix}$$

définie par Mac Robert [Proc. Roy. Soc. Edinburgh 58 (1937), 1–13] l'auteur donne des représentations sous forme d'intégrales définies, ainsi que pour des fonctious E^*, E_1, $E_1{}^*$, qu'il définit et qui sont proches de la fonction E. (Les fonctions E^*, E_1, $E_1{}^*$ ou l'une de leurs dérivées successives se retrouvent sous le signe intégral.)

R. Campbell (Caen)

7116:

Singh, V. N. A further note on the partial sums of certain basic bilateral hypergeometric series. Gaṇita 8 (1957), 71–79.

The results referred to in the title are obtained by using a transformation connecting two terminating ${}_{10}\Psi_{10}$ series. A bilateral extension of Watson's transformation of a well-poised ${}_8\Phi_7$ into a Saalschützian ${}_4\Phi_3$ is also given.

N. D. Kazarinoff (Ann Arbor. Mich.)

7117:

Singh, V. N. The basic analogues of identities of the Cayley-Orr type. J. London Math. Soc. 34 (1959), 15–22.

L'auteur rappelle le théorème de Cayley: Si

$$(1-z)^{\alpha+\beta-\gamma}F(2\alpha, 2\beta; 2\gamma, z)=\sum a_n z^n$$

alors

$$F(\alpha, \beta; \gamma+\tfrac{1}{2}; z)F(\gamma-\alpha, \gamma-\beta; \gamma+\tfrac{1}{2}; z)=\sum\frac{(\gamma)_n}{(\gamma+\frac{1}{2})_n}a_n z^n$$

et les conséquences qu'en ont tirées Clausen, Edwardes, Watson et Whipple, et surtout Orr (1899). Il donne ici une généralisation des identités du type Orr-Cayley relatives à des quantités dites Saalschützian et de la forme:

$${}_r\Phi_s\begin{bmatrix}x^{a_1}, x^{a_2}, \cdots, x^{a_r}, z\\ x^{b_1}, x^{b_2}, \cdots, x^{b_s}\end{bmatrix}=\sum\frac{(x^{a_1}; n)(x^{a_2}; n)\cdots(x^{a_r}; n)}{(x; n)(x^{b_1}; n)\cdots(x^{b_s}, n)}z^n;$$

$$(q^\alpha; n)=(1-q^\alpha)(1-q^{\alpha+1})\cdots(1-q^{\alpha+n-1}).$$

R. Campbell (Caen)

7118:

Agarwal, Nirmala. Certain basic hypergeometric identities of the Cayley-Orr type. J. London Math. Soc. 34 (1959), 37–46.

Dans cette étude, l'auteur redonne par une autre méthode (introduction d'un opérateur différentiel, d'ailleurs très simple) les identités (du reste compliquées) du type Orr-Cayley données par Singh [compte rendu précédent].

R. Campbell (Caen)

7119:

Al-Salam, W. A. Some generating functions for the Laguerre polynomials. Portugal. Math. 17 (1958), 49–52.

Three generating functions for $L_n{}^\beta(x)$ are given. They are

$$(t^2-tx)^{-\alpha/2}e^{2t}I_\alpha[(t^2-tx)],$$

$$(1-t)^{-1}(-x/2)^{-\alpha}e^{tx/(t-1)}I_\alpha[tx/(t-1)]t^{-\alpha/2},$$

$$(t^2x^2-tx^2)^{-(\alpha+\beta)/2}e^{2tx}I_{\alpha+\beta}[2x(t^2-t)^{\frac{1}{2}}].$$

N. D. Kazarinoff (Ann Arbor, Mich.)

7120:

Carlitz, Leonard. Some biorthogonal q-polynomials in two variables. Boll. Un. Mat. Ital. (3) 13 (1958), 555-557. (Italian summary)

In an earlier paper [Duke Math. J. 25 (1958), 355–364; MR 20 #2480] the author proved an orthogonal property of basic Hermite polynomials derived from a positive definite quadratic form. In this paper he investigates corresponding polynomials derived from the indefinite form $2xy$. He finds an orthogonal property involving a weight function which is again reminiscent of a theta function except that the summation indices run only through non-negative integers.

A. Erdélyi (Pasadena, Calif.)

7121:

Franklin, J. N. An enveloping series for the zeta function. Nederl. Akad. Wetensch. Proc. Ser A. 61 = Indag. Math. 20 (1958), 505-507.

Let $\zeta(s, a)$ denote the generalized zeta function [E. T. Whittacker and G. N. Watson, *A course of modern analysis*, 4th ed., University Press, Cambridge, 1952; p. 265] and let B_n denote the nth Bernoulli number. The author shows that

$$\sum_{n=1}^{\infty}(-1)^{n-1}[(2n)!]^{-1}B_n s(s+1)\cdots(s+2n-2)a^{1-s-2n}$$

is an enveloping series [J. G. van der Corput, *Asymptotic expansions, II*, Dept. of Math., Univ. of California, Berkeley, California, 1955; MR 17, 1201] for the function $\zeta(s, a)-(s-1)^{-1}a^{1-s}-\frac{1}{2}a^{-s}$, valid for positive integers a and $\operatorname{Re} s>-1$. Modified results are also obtained for different complex values of s.

C. A. Swanson (Vancouver, B.C.)

ORDINARY DIFFERENTIAL EQUATIONS

See also 7391, 7402.

7122:

Miu, Ion M. A new method for determining a particular integral of a linear non-homogeneous differential equation. Gaz. Mat. Fiz. Ser. A. 6 (1957), 303-307. (Romanian)

This is an iterative process which the author claims to be new.

R. Blum (Saskatoon, Sask.)

7123:

Banditch, I. Sur l'intégration de deux équations différentielles importantes non-linéaires de deuxième ordre. Acad. Roy. Belg. Bull. Cl. Sci. (5) 44 (1958), 702–707.

The author shows that the differential equations $y''+f(x)y'=\varphi(x)y^n$ and $y''+f(x)y'=\varphi(x)e^y$ are solvable by quadratures only when $\varphi(x)=\text{const}\cdot\exp(-2\int f(x)dx)$, and gives the explicit solutions. *S. Katz* (New York, N.Y.)

7124:

Saito, Tosiya. On Fuchs' relation for the linear differential equation with algebraic coefficients. Kōdai Math. Sem. Rep. 10 (1958), 101–104.

Consider a homogeneous linear ordinary differential equation of order n. If the coefficients are rational functions and the equation is Fuchsian with m singular points, then the sum S of all the roots of the m fundamental equations satisfies the Fuchs relation $S=n(n-1)\times(m/2-1)$. The author permits the coefficients to be algebraic functions, rational on a Riemann surface of genus p, and establishes the generalized Fuchs relation $S=n(n-1)(m/2+p-1)$. *E. R. Kolchin* (New York, N.Y.)

7125:

Wintner, Aurel. On Riccati's resolvent. Quart. Appl. Math. 14 (1957), 436–439.

Let $x=x(t)$ be a (real) solution of a (real) linear system (1) $x'=A(t)x$, where $A(t)$ is a continuous n by n matrix. Let $r=|x|$, $e=x/r$. Then $e(t)$ satisfies the non-linear system (2) $e'=[A(t)-(e\cdot A(t)e)I]e$, where I is the unit matrix and the dot denotes scalar multiplication, and $r(t)$ satisfies (3) $(\log r)'=e\cdot A(t)e$. Conversely, if $e=e(t)$ is a solution of (2) satisfying $e(t_0)=1$ and $r(t)$ is given by (3), then $e(t)\equiv 1$ and $x(t)=r(t)e(t)$ is a solution of (1). For solutions of (2) satisfying $e(t)\equiv 1$, (2) can be reduced to a system of $n-1$ equations. Thus (2) is the analogue of Riccati's equation in the case $n=2$.

P. Hartman (Baltimore, Md.)

7126:

Feller, William. Sur une forme intrinsèque pour les opérateurs différentiels du second ordre. Publ. Inst. Statist. Univ. Paris 6 (1957), 291–301.

A brief survey, without proofs, of the author's recent work on abstract second order differential operators, the semigroups they generate, and their application to various classical problems (particularly the vibrating string). *H. Mirkil* (Hanover, N.H.)

7127:

Agudo, F. R. D.; et Wolf, František. Propriétés spectrales des équations différentielles non-autoadjointes. Atti Acad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 24 (1958), 643–645.

The authors consider perturbations of $A=d^{2n}/dx^{2n}$ by an operator of order $2n-1$ with coefficients in $L^2(-\infty,\infty)$. They then apply the concept of A-compactness of the perturbing operator to show that the perturbed operator has a real spectrum except for a discrete set of non-real eigenvalues. *R. R. Kemp* (Kingston, Ont.)

7128:

McLeod, J. B. Two expressions for the distribution of eigenvalues. Proc. Roy. Soc. London Ser. A 249 (1959), 513–517.

The paper is concerned with the distribution of the eigenvalues of

$$\frac{d^2\psi}{dr^2}+\left\{\lambda-q(r)-\frac{l(l+1)}{r^2}\right\}\psi=0\quad(0<r<\infty)$$

where $l=0, 1, 2, \cdots$.

In 1937, E. C. Kemble and R. E. Langer showed that these eigenvalues $\lambda_0, \lambda_1, \lambda_2, \cdots$ are roots of equations of the form

$$\int_{R_1}^{R_2}\left\{\lambda_m-q(r)-\frac{(l+\frac{1}{2})^2}{r^2}\right\}^{\frac{1}{2}}dr=(m+\tfrac{1}{2})\pi+\delta$$

where R_1 and R_2 are the zeros of the integrand and δ is small when m is large. E. C. Titchmarsh [Proc. Roy. Soc. London Ser. A 245 (1958), 147–155; MR 20 #2530] proved that the λ_m are the roots of equations of the form

$$\int_0^{p_m}\{\lambda_m-q(r)\}^{\frac{1}{2}}dr=(\tfrac{1}{2}l+m+\tfrac{3}{4})\pi+\delta'$$

where $p(\lambda)$ is the inverse function to $q(r)$ and $p_m=p(\lambda_m)$ and where δ' is small when m is large.

The present paper shows that the two results are consistent under the following conditions on $q(r)$: (a) $q(r)$ is continuously differentiable with $q(r)\to+\infty$ as $r\to\infty$; $q'(r)\geqq 0$; (b) $q'(r)$ is a steadily increasing function.

E. T. Copson (St. Andrews)

7129:

Wintner, Aurel. A comparison theorem for Sturmian oscillation numbers of linear systems of second order. Duke Math. J. 25 (1958), 515-518.

Consider the real differential system $x''+F(t)x=0$ where x is a vector in n-space and $F(t)$ a continuous matrix function of t, defined on a (possibly infinite) interval θ. The "oscillation number" of this problem, denoted by F_θ, is then defined as the least number such that no solution vector $x(t)\not\equiv 0$ will vanish at more than F_θ values of $t\in\theta$.

Let $f(t)$ be the greatest eigenvalue of the Hermitian part of $F(t)$ and consider the 1-dimensional problem $u''+f(t)u=0$ over θ, with oscillation number f_θ. The author then shows that $F_\theta\leqq f_\theta$. The proof proceeds by deriving a differential equation for the length $r(t)$ of $x(t)$ and applying the Sturm comparison theorem. [For the validity of this procedure see N. Levinson's paper, reviewed below]. A generalization is also given for the case of a system of the form $x''+G(t)x'+F(t)x=0$.

R. Bott (Ann Arbor, Mich.)

7130:

Levinson, Norman. Remark about Wintner's comparison theorem. Duke Math. J. 25 (1958), 519-520.

This paper adds a note to the preceding paper by A. Wintner [see #7129 above], clarifying the applicability of the classical Sturm comparison theorem, which Wintner used. *R. Bott* (Ann Arbor, Mich.)

7131:

Cuciuc, M. On some asymptotic formulae. Lucrarile Inst. Petrol Gaze Bucureşti 4 (1958), 239-251. (Romanian. Russian and English summaries)

7132:

Seifert, George. The asymptotic behavior of solutions of pendulum-type equations. Ann. of Math. (2) 69 (1959), 75–87.

Consider the system of equations (1): $\dot\theta=z$, $\dot z=g(\theta)-\alpha f(\theta)z+p(t)$, $\alpha>0$, $f(\theta)>0$; $f(\theta+2\pi)=f(\theta)$, $g(\theta+2\pi)=g(\theta)$;

$p(t)$, $g''(\theta)$, $f'(\theta)$ are continuous everywhere; there exists a constant k such that $|p(t)|<k$ for all t; and each of the equations $g(\theta)=\pm k_1$, $0\leqq k_1\leqq k$, has simple roots. For (2) $p(t)=0$, $\int_0^{2\pi}g(\theta)d\theta>0$, the author has previously considered system (1), and even more general systems [see, e.g., Proc. Amer. Math. Soc. **7** (1956), 1082–1084; Contributions to the theory of nonlinear oscillations, vol. 3, pp. 1–16, Annals of Math. Studies no 36, Princeton Univ. Press, 1956; Z. Angew. Math. Phys. **7** (1956), 238–247; MR **18**, 483, 305; **17**, 1207]. If (2) is satisfied, it is known there exists an $\alpha_m>0$ such that each solution $(\theta(t), z(t))$ of (1) approaches some constant solution $(\theta_i, 0)$ of (1) as $t\to\infty$ for $\alpha>\alpha_m$, and, in the last quoted paper, an explicit expression was given for α_m. In the first part of the present paper, an explicit expression for α_m is given when $p(t)=0$ and $\int_0^{2\pi}g(\theta)d\theta\neq0$. In the second part of the paper, the case where $p(t)\neq0$ is discussed and it is shown that there is a region D in the cylindrical phase space such that each solution $(\theta(t), z(t))$ of (1) is in D for $t>t_0$, where t_0 depends upon the particular solution. In particular, if $p(t)$ is periodic with period τ, then there is a periodic solution of (1) in D of period τ. The stability properties of the solutions are also discussed. *J. K. Hale* (Baltimore, Md.)

7133:

✱Bass, Robert W. On non-linear repulsive forces. Contributions to the theory of nonlinear oscillations, Vol. IV, pp. 201–211. Annals of Mathematics Studies, no. 41. Princeton University Press, Princeton, N. J., 1958. ix+211 pp. $3.75.

Let x, x', f be Euclidean n-vectors. Let $f(t, x, x')$ be continuous for $t\geqq0$ and all x, x' with the properties that the scalar product $x\cdot f(t, x, x')$ is non-negative for $t\geqq0$ and all x, x' and that, for every $R>0$, there exists a constant C such that $|f(t, x, x')|\leqq C(1+|x'|)$ for $0\leqq t$, $|x|\leqq R$ and all x'. The main conclusion is that if x_0 is arbitrary, then there exists at least one x_0' such that the initial value problem (1) $x''=f(t, x, x')$, $x(0)=x_0$, $x'(0)=x_0'$ has at least one solution on $0\leqq t<\infty$ such that $r=|x(t)|^2$ satisfies $r\geqq0$, $r'\leqq0$, $r'\geqq0$ (in particular, $x(t)$ is bounded). The proof is a refinement of a procedure of A. Kneser [J. Reine Angew. Math. **116** (1896), 178–212] and of Wintner [Amer. J. Math. **71** (1949), 362–366; MR **10**, 711]. *P. Hartman* (Baltimore, Md.)

7134:

Ševelo, V. M. On the approximate investigation of oscillatory systems. Dopovidi Akad. Nauk Ukraïn RSR **1958**, 609–612. (Ukrainian. Russian and English summaries)

Išlinsky's approximate method of investigating oscillatory systems [Akad. Nauk Ukrain. RSR. Prikl. Meh. **2** (1956), 152–158; MR **18**, 305; see p. 157] is extended to oscillatory systems governed by ordinary differential equations of the third order with variable coefficients. An approximate criterion is obtained for the damping oscillations governed by the equation $a_0\dddot{\varphi}+a_1\ddot{\varphi}+a_2\dot{\varphi}+a_3\varphi=0$, where $a_i=a_i(t)$, $i=0, \cdots, 3$. Using the phase method, one inequality for φ is deduced; as a special case of this one obtains Routh's inequality for ordinary equations of the third order with constant coefficients and Leonov's criterion for equations of the second order with variable coefficients [Prikl. Mat. Meh. **10** (1946), 575–580,]. As an example, a damping criterion is found for the damping of forces in an imponderable thread of variable length, having a load attached to the end, with a linear law of relaxation and aftereffect. *D. P. Rašković* (Belgrade)

7135:

Faure, Robert. Existence et stabilité des solutions périodiques de certains systèmes de n équations différentielles à coefficients périodiques; cas où $p\leqq n$ fonctions associées sont identiquement nulles. C. R. Acad. Sci. Paris **248** (1959), 520–523.

The author considers systems of the form $dx/dt=\lambda f(x, t)$ where $x=(x_1, x_2, \cdots, x_n)$ and $f=(f_1, f_2, \cdots, f_n)$ are real vectors, λ and t are real parameters and f is periodic in t of period T. It is supposed for some positive integer $p\leqq n$ that $\int_0^T f_i(\alpha, t)dt=0$, $1\leqq i\leqq p$, for all α in some domain D of x space. The author first considers the case of a linear (homogeneous) system and develops the form of a basis system of solutions. The case of a nonlinear f is treated by means of a development about an α in D. Conditions generalizing those of Haag [Bull. Sci. Math. (2) **70** (1946), 21–36; MR **8**, 273] are given for the stability of solutions about such points α. *C. E. Langenhop* (Ames, Iowa)

7136:

Zubov, V.I. Über die Stabilitätsbedingungen in einer endlichen Zeitstrecke und über die Bestimmung der Länge des Intervalls. Bul. Inst. Politehn. Iaşi (N.S.) **4(8)** (1958), 69–74. (Russian. German and Romanian summaries)

The system dealt with is the n-vector system

$$\dot{x}=P(t)x+X(x, t),$$

where $P(t)$ is bounded continuous, and the components of X are convergent power series in the x_i beginning with terms of degree $\geqq2$ whose coefficients are continuous and bounded functions of t. The problem under discussion has already been dealt with by Kamen [Akad. Nauk SSSR. Prikl. Mat. Meh. **17** (1953), 529–540; MR **15**, 795], Lebedev [ibid. **18** (1954), 75–94, 139–148; MR **16**, 132] and Kamenkov and Lebedev [ibid. **18** (1954), 512; MR **16**, 361], but they did not determine accurately the time interval in which the origin is stable. In the present note the author determines necessary and sufficient conditions of stability for a finite interval and also gives a method for computing the length of the interval.

The author uses the following definition of stability: Given a definite positive quadratic form, $V(x)$, the origin is stable relative to V on the time interval τ if

$$V(x(t, t_0, x_0))<A$$

for $t\in[t_0, t_0+\tau]$ and $V(x_0)\leqq A$, where A is sufficiently small. Theorem 1: A sufficient condition for the stability just stated whatever X for suitable small τ and A is that the characteristic roots of $P(t_0)$ have negative real parts. The calculation of A and τ is outlined. Theorem 2: If not all of the characteristic roots of $P(t_0)$ have negative real parts, th. 1 does not hold. *S. Lefschetz* (Mexico, D.F.)

7137:

Atkinson, F. V. On stability and asymptotic equilibrium. Ann. of Math. (2) **68** (1958), 690–708.

For a vector $x=(x_1, \cdots, x_p)$, let $|x|=\max|x_r|$. Consider a real system of $n+m$ differential equations

$$(1)\qquad y'=h(t, y, z)+g(t, y, z),\quad z'=j(t, y, z),$$

where y, h, g are n-vectors and z, j are m-vectors. In (1), let h, g, $j\in(C^0, \mathrm{Lip}, \mathrm{Lip})$ for $t\geqq0$, $|y|\leqq D'$, $|z|<\infty$. Let $0\leqq k\leqq n$ and assume that $y_rh_r(t, y, z)>0$ or <0 for $y_r\neq0$ according as $1\leqq r\leqq k$ or $k<r\leqq n$. For given positive numbers d, D such that $d<D/n$, $D<D'$, assume that

$$(2)\qquad \int_u^v|g_r(t, y(t), z(t))dt|<\int_u^v|h_r(t, y(t), z(t))|dt+d$$

holds for all solutions $y=y(t)$, $z=z(t)$ of (1) satisfying $|y(t)|\leq D$ for $(0\leq)u\leq t\leq v$. Then, for any given initial $z(0)$, there exists an $(n-k)$-parameter family of solutions $y(t)$, $z(t)$ of (1) satisfying $|y(t)|\leq D$ for $0\leq t\leq\infty$. This is the main theorem of the paper. It is proved by showing that if the assertion is false, the "flow" leads to an impossible mapping. This argument is related to those of Hartman and Wintner [Amer. J. Math. **77** (1955), 692–724; MR **17**, 485] and of Ważewski. The main theorem has corollaries avoiding the awkward condition (2) involving solutions of (1) and generalizing many known results on the asymptotic integration of linear and non-linear systems. *P. Hartman* (Baltimore, Md.)

7138:

Corduneanu, C. Sur la stabilité conditionnelle par rapport aux perturbations permanentes. Acta Sci. Math. Szeged **19** (1958), 229–236.

By the use of a previous result for $n=1$ [Rev. Math. Pures Appl. **2** (1957), 497–500; MR **20** # 1042] and the Schauder-Tychonoff fixed point theorem the following statement is proved concerning the behavior of solutions of systems of the form (*) $dx_i/dt=f_i(t, x_1, \cdots, x_n)+R_i(t, x_1, \cdots, x_n)$, $i=1, \cdots, n$. Suppose that the functions f_i are continuous for $t\geq t_0$, $|x_1|, \cdots, |x_n|\leq H$, $i=1, \cdots, n$. Suppose that each f_i has continuous partial derivatives $\partial f_i/\partial x_i$ and that either (A) $-M\leq\partial f_i/\partial x_i\leq -m<0$, or (B) $0<m\leq\partial f_i/\partial x_i\leq M$, $i=1, \cdots, n$. Suppose that $f_1(t, 0)=0$ and $|f_i(t, x_1, \cdots, x_{i-1}, 0, \cdots, 0)|\leq L(|x_1|+\cdots+|x_{i-1}|)$, $i=2, \cdots, n$. The functions R_i are supposed to be continuous and bounded, with $QR\leq H$ for some $Q>0$, where $R=\sup|R_i|$, $t\geq t_0$, $|x_1|, \cdots, |x_n|\leq H$. Then the system (*) has at least one solution $x(t)$ existing in $[t_0, \infty)$, $x=(x_1, \cdots, x_n)$, such that $P\sum|x_i(t)|\leq H-QR$, where P is a constant depending only on L and m, and $\sum$ ranges over all i for which (A) holds. *L. Cesari* (Baltimore, Md.)

PARTIAL DIFFERENTIAL EQUATIONS

See also 7196.

7139:

de Vito, Luciano. Su una limitazione degli autovalori relativi a problemi al contorno per equazioni differenziali a derivate parziali di ordine $2n$. Boll. Un. Mat. Ital. (3) **13** (1958), 319–326. (English summary)

Considering a class of self-adjoint partial differential operators E whose resolvents are assumed to be completely continuous, the author gives a number of inequalities of the type

$$\sum \lambda_n^s|(f, \varphi_n)|^2\leq\|E^{s/2}f\|^2$$

for the eigenvalues λ_n of E.

J. T. Schwartz (Berkeley, Calif.)

7140:

Pliś, A. The characteristic equation for partial differential equations of the first order. Colloq. Math. **6** (1958), 223–226.

If the partial differential equation $z_x=f(x, y, z, z_y)$ has a solution $z(x, y)$, if f is of class C^1 and if f_q satisfies a Lipschitz condition of order 1 and is monotone in q, $q=z_y$, then the author proves that every solution of the ordinary differential equation $d\omega/dx=-f_q(x, \omega, z(x, \omega), z_y(x, \omega))$, $y=\omega(x)$, together with $z(x, \omega)$ and $z_y(x, \omega)$ satisfies the usual characteristic differential system.

M. Steinberg (Los Angeles, Calif.)

7141:

Bellman, Richard. On a Liouville transformation for $u_{xx}+u_{yy}\pm a^2(x, y)u=0$. Boll. Un. Mat. Ital. (3) **13** (1958), 535–538. (Italian summary)

It is shown that under the assumption that $\log a(x, y)$ is harmonic we can find a change of independent variable which reduces the equation to one with constant coefficients. *Author's summary*

7142:

Friedman, Avner. On two theorems of Phragmén-Lindelöf for linear elliptic and parabolic differential equations of the second order. Pacific J. Math. **7** (1957), 1563–1575.

The theorems of Phragmén and Lindelöf for analytic functions give information on the behavior of such a function at an isolated boundary point. Generalizations to solutions of elliptic partial differential equations have been given by D. Gilbarg [J. Rational Mech. Anal. **1** (1952), 411–417; MR **14**, 279], by E. Hopf [ibid. **1** (1952), 419–424; MR **14**, 279] and by J. B. Serrin [ibid. **3** (1954), 395–413; MR **16**, 42]. In the present work the author presents several new extensions, both to elliptic and to parabolic equations in two and in higher dimensions. Typical theorems follow.

Let $u(x)$, $x=(x_1, \cdots, x_n)$, be a solution of $a_{ij}(x)(\partial^2u/\partial x_i\partial x_j)+b_i(x)(\partial u/\partial x_i)=0$ in an unbounded domain D which lies interior to the n-dimensional cone K_β with angular opening β, where $\frac{1}{2}\pi<\beta\leq\pi$ for $n\geq3$, $0<\beta\leq2\pi$ for $n=2$. Suppose that $a_{ij}(x)\to\delta_{ij}$ as $x\to\infty$ and that $b_i(x)\to0$ suitably quickly as $x\to\infty$. Then if, for some $\eta>0$, $\lim_{r\to\infty} r^{\eta-\pi/\beta}\sup_{x\in D,|x|=r} u(x)=0$, and if $u(x)\to0$ on ∂D as $x\to\infty$, it follows that $u(x)\to0$ uniformly in D as $x\to\infty$.

Let $u(X)=u(x, t)$ be a solution of $a_{ij}(X)(\partial^2u/\partial x_i\partial x_j)+b_i(X)(\partial u/\partial x_i-\partial u/\partial t)=0$ in an unbounded domain D contained in the half space $t>0$. Under suitable conditions on the a_{ij}, b_i, if $\lim_{R\to\infty} R^2\sup_{x\in D,|x|=R} u(X)=0$ and if $u(X)\to0$ on ∂D as $X\to\infty$, then $u(X)\to0$ uniformly in D as $X\to\infty$.

Proofs use the comparison method as developed in the above references, and depend on the construction of suitable comparison functions.

R. Finn (Stanford, Calif.)

7143:

Zlamál, Miloš. Über die erste Randwertaufgabe für eine singulär perturbierte elliptische Differentialgleichung. Czechoslovak Math. J. **7** (82) (1957), 413–417. (Russian summary)

Suppose that $u(x, \varepsilon)$ is defined for x in an n-dimensional bounded region G, and that u satisfies

$$\text{(1)}\quad -a(x, \varepsilon)\Delta u+u=F(x, \varepsilon)\quad (a>0,\ 0<\varepsilon<\varepsilon_0,\ x\in G)$$

under the condition $u=f(\varepsilon)$ on the boundary of G. Under certain conditions on a, F, and f, it is shown that if

$$\text{(2)}\quad \lim_{\varepsilon\to0+} a(x, \varepsilon)=0,\quad \lim_{\varepsilon\to0+} F(x, \varepsilon)=F_0(x)$$

uniformly on $\bar G$, then

$$\text{(3)}\quad \lim_{\varepsilon\to0+} u(x, \varepsilon)=F_0(x)$$

uniformly in every closed subregion of G. An estimate of the rate of convergence in (3) is also given. These results simplify and extend a previous result of Morgenstern [J. Rational Mech. Anal. **5** (1956), 203–216; MR **17**, 1211].

J. Elliott (New York, N.Y.)

7144:

Miles, E. P., Jr.; and Williams, Ernest. The Cauchy problem for the damped wave equation with polyharmonic initial conditions. Portugal. Math. **17** (1958), 53–57.

Gli A. danno una formula risolutiva per il problema di Cauchy relativo all'equazione

$$\left[\nabla^2-\left(\frac{\partial^2}{\partial t^2}-\kappa^2\right)\right]u(x_1,\cdots,x_r,t)=0$$

quando i dati iniziali:

$$u(x_1,\cdots,x_r,0)=F(x_1,\cdots,x_r),$$
$$u_t(x_1,\cdots,x_r,0)=G(x_1,\cdots,x_r)$$

sono funzioni poliarmoniche di ordine f e g rispettivamente. Si ha precisamente

$$u=\sum_{n=0}^{f-1}\nabla^{2n}F\cdot u_n+\sum_{n=0}^{g-1}\nabla^{2n}G\cdot v_n,$$

con

$$u_0=\cosh \kappa t,\quad u_1=\kappa t\sinh \kappa t/2\kappa^2,$$
$$v_0=\sinh \kappa t,\quad v_1=(-\sinh \kappa t+t\cosh \kappa t)/2\kappa^2,$$
$$u_n=\frac{1}{\kappa^{2n}}\sum_{j=1}^{n}A_j{}^nE_j,\quad v_n=\frac{1}{\kappa^{2n}}\sum_{j=1}^{n}B_j{}^nO_j\quad(n\geqq 2)$$

dovè

$$A_j{}^n=\frac{(-1)^{j+n}(2n-j-1)!}{(2n)!!2^{n-j}(n-j)!(j-1)!};$$
$$B_j{}^n=(2n+2)A_{j+1}{}^{n+1};$$
$$E_{2m}=(\kappa t)^{2m}\cosh \kappa t,\quad E_{2m+1}=(\kappa t)^{2m+1}\sinh \kappa t;$$
$$O_{2m}=(\kappa t)^{2m}\sinh \kappa t,\quad O_{2m+1}=(\kappa t)^{2m+1}\cosh \kappa t.$$

Formule analoghe si hanno anche per gli analoghi problemi di Cauchy per l'equazione

$$(\nabla^2-\kappa^2)u=0;\quad (\nabla^2+\kappa^2)u=0.$$

E. Magenes (Genova)

7145:

Kampé de Fériet, Joseph. Équation de la chaleur et polynomes d'Hermite. C. R. Acad. Sci. Paris **248** (1959), 883–887.

Let the Hermite polynomial $H_n(x)$ be defined as the nth partial derivative of the kernel $\exp[hx-\frac{1}{2}h^2]$ with respect to h. The author recalls that the polynomial $K_n(x, t)$ defined as

$$K_n(x,t)=(i(2t)^{\frac{1}{2}})^nH_n\left(\frac{x}{i(2t)^{\frac{1}{2}}}\right)$$

is a solution of the heat equation $u_{xx}=u_t$. He points out that these polynomials are closely related to the group (or semi-group) associated with the heat equation. Let the operator T_t be defined on the space E of entire functions, endowed with a suitable Fréchet topology, by the equations

$$T_t[x^n]=K_n(x,t).$$

Then $K_n(x, s+t)=T_s[T_t(x^n)]$ for all s and t. The operation may be extended to suitable subsets of E by setting

$$T_t\{\sum A_nx^n\}=\sum A_nK_n(x,t).$$

If, for instance, F_τ is the subset of E characterized by

$$\lim\sup((n!)^{\frac{1}{2}}A_n)^{1/n}=(2\tau)^{-\frac{1}{2}},$$

then $f(x)\in F_\tau$ implies that $F_t[f]\in F_{\tau-t}$ for $|t|<\tau$, $u(x,t)\equiv T_t[f](x)$ satisfies the heat equation in the strip $|t|<\tau$ of the (x, t)-plane, and $\lim_{t\to 0}u(x,t)=f(x)$. There are functions of F_τ for which the solutions have singular points on $t=\tau$. If $f(x)=f(-x)$ and $f(x)\in F_\tau$, then the series cannot converge absolutely for $t>\tau$.

E. Hille (New Haven, Conn.)

7146:

Rosenbloom, P. C.; and Widder, D. V. A temperature function which vanishes initially. Amer. Math. Monthly **65** (1958), 607–609.

It is known that $\partial^2u/\partial x^2=\partial u/\partial t$ has solutions $u(x,t)$ of class C^∞ on the whole plane satisfying $u(x,0)=0$. The authors give a simple example of this type.

P. Hartman (Baltimore, Md.)

7147:

Friedrichs, K. O. Symmetric positive linear differential equations. Comm. Pure Appl. Math. **11** (1958), 333–418.

La théorie développée ici généralise celle, dûe à l'A., des équations "hyperboliques symétriques" [Comm. Pure Appl. Math. **7** (1954), 345–392; MR **16**, 44]. Conçue principalement en vue de l'étude des équations de type mixte, elle s'applique à une très large classe d'équations aux dérivées partielles, englobant toutes les équations classiques du deuxième ordre. Son originalité réside dans le fait qu'elle n'utilise pas le caractère elliptique ou hyperbolique des équations, et qu'elle est fondée uniquement sur les propriétés des opérateurs linéaires du premier ordre (presque permutabilité, inégalités relatives aux intégrales d'énergie, et identité des extensions faible et forte) établie antérieurement par l'A. [Trans. Amer. Math. Soc. **55** (1944), 132–151; MR **5**, 188].

Le système étudié est supposé mis sous la forme $Ku=f$, où $u=\{u_\nu\}$ ($\nu=1, 2, \cdots, k$) désigne une fonction vectorielle inconnue, $f=\{f_\nu\}$ une fonction vectorielle donnée, et $K=\sum\alpha^\rho(\partial/\partial x^\rho)+\gamma$ ($\rho=1, 2, \cdots, m$) un opérateur différentiel linéaire du premier ordre, défini par la donneé de $m+1$ matrices symétriques $\alpha^\rho=\{\alpha^\rho{}_{\lambda\nu}\}$, $\gamma=\{\gamma_{\lambda\nu}\}$; la partie symétrique de la matrice $\kappa=\gamma-\alpha^\rho\partial/\partial x^\rho$ est supposée définie positive. Toutes ces fonctions sont supposées définies sur une variété finie $\mathscr{R}$ à m dimensions, à bord régulier $\mathscr{B}$, localement défini par $y=x^m=0$. Les conditions de régularité imposées à $\mathscr{R}$ et aux données, autres que f, ne sont pas précisées, mais il semble suffire qu'elles soient de classe C^2. Les formules de transformation $u\to\tilde{u}$ correspondant à un changement de coordonnées locales $x\to\tilde{x}$ sont définies de façon à conserver la forme différentielle $u\cdot vdx=\sum u_\lambda v_\lambda dx^1\wedge\cdots\wedge dx^n$, et on pose $(u,v)=\int_{\mathscr{R}}u\cdot vdx$, ce qui définit un espace de Hilbert $\mathscr{H}$. De même le produit scalaire $(u,v)_{\mathscr{B}}=\int_{\mathscr{B}}u\cdot vdx/dy$ définit un espace de Hilbert $\mathscr{H}_{\mathscr{B}}$. Enfin les conditions frontières imposées à u sont définies au moyen d'une décomposition $\beta=\beta_++\beta_-$ de la matrice $\beta=n_\lambda\alpha^\lambda$ (où n_λ désigne la normale à $\mathscr{B}$) telle que la partie symétrique de $\beta_+-\beta_-$ soit $\geqq 0$, et que chaque vecteur u se décompose en $u=u_++u_-$ de façon que $\beta_+u_-=\beta_-u_+=0$. La condition $\beta_-u=0$ est dite alors admissible à la frontière.

Toute solution forte de $Ku=f$, satisfaisant à $\beta_-u=0$ sur $\mathscr{B}$, vérifie une inégalité de la forme $\|u\|\leqq c\|f\|$, d'où unicité. L'unicité des solutions fortes pour l'opérateur adjoint K^* (qui se déduit de K par le changement de α en $-\alpha$ et de κ en la matrice transposée κ') entraîne l'existence de solutions faibles de $Ku=f$. L'identité des solutions faibles et fortes n'est pas établie directement [une étude de R. S. Phillips et P. D. Lax sur ce sujet est annoncée], mais établie sous l'hypothèse que f est fortement différentiable. Pour cela, l'A. introduit un système $D=\{D_\sigma\}$ ($\sigma=0, 1, 2, \cdots, m$) d'opérateurs différentiels linéaires du premier ordre (D_0 étant l'identité) complet en chaque point intérieur à $\mathscr{R}$, mais n'impliquant sur $\mathscr{B}$

aucune dérivée normale, et satisfaisant à des relations de la forme $[D_\sigma, K]=p_\sigma{}^\tau D_\tau+t_\sigma K$. L'espace $\mathscr{H}_1$ est celui des u tels que $Du \in \mathscr{H}$ et on utilise le "clipped Laplacian" $D^*\cdot D=\sum_\sigma D_\sigma{}^* D_\sigma$. L'étude précédemment faite pour K s'applique à l'opérateur $\underset{1}{K}$ défini par $\underset{1}{K}\underset{1}{u}=\{Ku_\sigma+p_\sigma{}^\tau u_\tau\}$ (où $\underset{1}{u}=\{u_\sigma\}$ désigne un système de vecteurs), et qui satisfait à $(D-t)K=\underset{1}{K}D$. On lui associe une matrice M sur $\mathscr{B}$, telle que $(D-t^{\mathscr{B}})M=\underset{1}{M}D$ (où $\underset{1}{M}=-2\beta_-$). Sous réserve que $\underset{1}{K}$ et $\underset{1}{M}$ soient admissibles, on démontre que si $f\in\mathscr{H}_1$, l'équation $Ku=f$ admet une solution faible $u\in\mathscr{H}_1$, et par la méthode des noyaux régularisants, on établit que u est solution forte; enfin on se débarrasse des hypothèses supplémentaires relatives à $\underset{1}{K}$ et $\underset{1}{M}$ {sur ces derniers points la démonstration gagnerait à être plus explicite}.

L'extension aux variétés dont les bords présentent des arêtes est possible pour certaines conditions frontières (qui correspondent à $\beta_-\equiv 0$ sur certaines parties du bord, et à $\beta_-\equiv\beta$ sur d'autres), ce qui permet de déterminer des conditions frontières admissibles particulièrement significatives pour l'équation de Tricomi.

Enfin l'A. expose en détails comment on peut ramener au type étudié, au moyen de transformations convenables, certaines équations classiques. *J. Lelong* (Paris)

7148:

Yosida, Setuzô. Hukuhara's problem for hyperbolic equations with two independent variables. II. Quasi-linear case. Proc. Japan Acad. 34 (1958), 466–470.

In Part I of this report [same Proc. **34** (1958), 319–324; MR **20** #4095] the author explained the concept of Hukuhara's problem and proved that it is correctly posed for semi-linear hyperbolic systems with two independent variables. He now proves the same results for a quasi-linear hyperbolic system. *E. T. Copson* (St. Andrews)

7149:

Gheorghiu, O. E. Une classe particulière d'équations linéaires aux dérivées partielles de 3e ordre, à trois variables indépendantes. Acad. R. P. Romîne. Baza Cerc. Şti. Timişoara. Stud. Cerc. Şti. **3** (1956), no. 1–2, 17–27. (Romanian. Russian and French summaries)

The independent variables are interpreted as the polar coordinates r, θ, φ in S_3 and the coefficients are assumed to be entire series in r^3. Using a method given by N. Cioranesco the author investigates the conditions for the existence of a particular solution of a certain form of the given equation. *R. Blum* (Saskatoon, Sask.)

7150:

Pini, Bruno. Sulle equazioni paraboliche lineari del quarto ordine. I, II. Rend. Sem. Mat. Univ. Padova **27** (1957), 319–349, 387–410.

In these two papers the author presents an extensive treatment for the following normalized boundary-value problem:

$$(1)\quad \mathscr{L}[u]\equiv\frac{\partial^4 u}{\partial x^4}-2a\frac{\partial^3 u}{\partial x^2\partial y}+\frac{\partial^2 u}{\partial y^2}+\sum_{i=1}^{3}a_i\frac{\partial^i u}{\partial x^i}+b\frac{\partial^2 u}{\partial x\partial y}+c\frac{\partial u}{\partial y}=f$$

in a region R: $0<x<1$; $0<y\leq h$, with u, $\partial u/\partial y$ prescribed on $x=0$ and u, $\partial u/\partial x$ prescribed on both $x=0$ and $x=1$. The a, a_i, b, c, f are given functions of position in the closed rectangle $\bar{R}$. Moreover, it is required that either $0<a\neq 1$ or $a\equiv 1$ throughout $\bar{R}$.

In the first paper, the case $a>1$ in $\bar{R}$, is studied in detail by first considering the reduced boundary-value problem for the reduced equation

$$(2)\quad \mathscr{L}_0[u]\equiv\frac{\partial^4 u}{\partial x^4}-2a_0\frac{\partial^3 u}{\partial x^2\partial y}+\frac{\partial^2 u}{\partial y^2}=f$$

(with $a_0\equiv$constant >1 in $\bar{R}$) and solving it through the construction of a fundamental solution. This leads to the construction of a parametrix when a_0 is replaced by the actual coefficient $a(x, y)$ and to a sequence of a priori estimates on functions represented by the parametrix kernel. Finally a solution for (1) is obtained in the usual manner through an integral equation of the Volterra type. The actual existence theorem states in this case: Let a, a_i, b, c, $f\in C^1(\bar{R})$, $0\neq a<1$, in $\bar{R}$; let the given data be derived from a function φ continuous with all its derivatives that enter in $\mathscr{L}(\varphi)$ in $\bar{R}$ and $\mathscr{L}(\varphi)$ Hölder continuous in R; then the boundary value problem has at least one solution which belongs to $C^2(\bar{R})$ and whose derivatives that enter in $\mathscr{L}$ are continuous in R. A uniqueness theorem is proved under more stringent assumptions on the coefficients of $\mathscr{L}$.

The second paper indicates a similar analysis for the other two cases $a>1$ in $\bar{R}$ and $a\equiv 1$ in $\bar{R}$. All three cases are unified by the actual construction of the unique solution for $\mathscr{L}_0$ with $a_0\equiv 1$. Then suitable Banach spaces are constructed on which the transformations $\mathscr{L}_\omega$, where $\omega=(a, a_i, b, c)$, furnish a family of continuous linear mappings, depending continuously on ω. The complete invertibility of $\mathscr{L}_\omega$ for $\omega_0=(1, 0, 0, 0)$ leads to a solution of $\mathscr{L}u=f$ for ω sufficiently close to ω_0. *C. R. DePrima* (Pasadena, Calif.)

7151a:

Pini, Bruno. Un problema di valori al contorno per un'equazione a derivate parziali del terzo ordine con parte principale di tipo composito. Rend. Sem. Fac. Sci. Univ. Cagliari **27** (1957), 114–135.

7151b:

Pini, Bruno. Su una equazione parabolica non lineare del quarto ordine. Rend. Sem. Fac. Sci. Univ. Cagliari **27** (1957), 136–168.

The author continues his study of boundary-value problems related to equations of the type discussed in the preceding review. [The notation and definitions of the preceding review will be employed.] In these two papers two limiting cases of equations whose normal form is given by (1) are investigated. The first paper deals with the equation which, in normal form, has the principal part (3) $\mathscr{L}_1[u]\equiv\partial^4 u/\partial x^4$, while the second deals with an equation with principal part (4) $\mathscr{L}_2[u]\equiv\partial^3 u/\partial x^2\partial y-\partial^2 u/\partial y^2$. The complete boundary value problems are then

$$(5)\quad \mathscr{L}_1[u]=-\frac{\partial u}{\partial y}+f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^3 u}{\partial x^3}\right)$$

in R, with u prescribed on $y=0$ and u, $\partial u/\partial x$ on $x=0$ and $x=1$; and

$$(6)\quad \mathscr{L}_2[u]=g\left(x, y, u\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x\partial y}\right)$$

in R, with u, $\partial u/\partial y$ prescribed on $y=0$ and u on $x=0$ and $x=1$.

Again the boundary value problems for the homogeneous equations: $\mathscr{L}_1[u]+\partial u/\partial y=0$ and $\mathscr{L}_2[u]=0$ are solved via fundamental solutions. A priori Hölder type estimates

are then obtained for solutions of the non-homogeneous equations. With these estimates in mind, natural Hölder continuity conditions on the non-linear parts of (5) and (6), respectively, lead to a contraction mapping for sufficiently small h and thus to a unique solution of the respective boundary-value problems. In the first paper, the equation $\mathscr{L}_1[u]=F(x, y, u, \partial u/\partial x, \partial^2u/\partial x^2, \partial^3u/\partial x^3, \partial u/\partial y)$, with $\partial F/\partial(\partial u/\partial y)<0$, is also treated.

C. R. DePrima (Pasadena, Calif.)

DIFFERENTIAL ALGEBRA

See 7155.

POTENTIAL THEORY

7152:

Kulshrestha, P. K. On existence of variation for a class of bounded harmonic functions. Gaṇita 8 (1957), 37–39.

Let $g(z, \zeta)$ denote the Green's function of a domain. The author considers the variation in $\sum g(z, \zeta_\nu)$ when ζ_ν are shifted. The hypothesis regarding (ζ_ν) is not clear.

S. M. Shah (Madison, Wis.)

7153:

Shapiro, Victor L. The divergence theorem for discontinuous vector fields. Ann. of Math. (2) 68 (1958), 604–624.

If $V=(A(X), B(X))$, $X=(x, y)$ is a vector field in a planar domain and div V is $\partial_xA+\partial_yB$, then for the validity of

$$\int_C Ady-Bdx=\iint_D \operatorname{div} Vdxdy,$$

where C is a rectifiable Jordan curve bounding D, the author gives the following criterion which in a certain direction [opened up by this reviewer] is in a sense definitive. Circumventing differentiation the author introduces

$$\operatorname{div}^* V=\limsup_{t\to 0}\,(\pi t^2)^{-1}\int_{C(x_0,t)} Ady-Bdx,$$

where $C(X_0, t)$ is the circumference of a circle with center X_0 and radius t, and similarly $\operatorname{div}_* V$ by lim inf. Now, the criterion is: (i) $V(X)$ is continuous in $D+C$, and (ii) L^2 on D, (iii) $\operatorname{div}^* V$ and $\operatorname{div}_* V$ are finite on D, and (iv) almost everywhere in D, $\operatorname{div}^* V=\operatorname{div}_* V=\operatorname{div} V$ and the latter is L^1 on D. — Also, both in (i) and (iii) a set Z of logarithmic capacity 0 may be excepted.

S. Bochner (Princeton, N.J.)

7154:

Leja, F. Sur certaines suites liées aux ensembles plans et leur application à la représentation conforme. Ann. Polon. Math. 4 (1957), 8–13.

Let E be an infinite, closed, and bounded set of points in the complex plane and let $a_1, a_2, a_3, \cdots$ be a sequence of points of E defined inductively in the following manner: a_1 is an arbitrary point of E and, given $a_1, a_2, \cdots, a_n$ $(n\geqq 1)$, the point a_{n+1} is any point of E for which

$$\prod_{k=1}^{n} |a_{n+1}-a_k|=\max_{z\in E} \prod_{k=1}^{n} |z-a_k|.$$

It is shown that the sequence $\{\prod_{k=1}^n |a_{n+1}-a_k|\}^{1/n}$ converges to the transfinite diameter $v(E)$ of E. Moreover, form the polynomials $P_n(z)=\prod_{k=1}^n (z-a_k)$ and normalize the functions $\{P_n(z)\}^{1/n}$ so that the quotient $\{P_n(z)\}^{1/n}/z$ is equal to 1 at infinity. Also, denote by D_∞ the unbounded component of the complement of E relative to the complex plane. It is shown that if $v(E)>0$, the sequence $\{P_n(z)\}^{1/n}$ converges in D_∞ to a single- or multiple-valued analytic function $P(z)$ which possesses a single-valued modulus in D_∞ and satisfies the relation

$$\log |P(z)|=G(z)+\log v(E),$$

where $G(z)$ is either the classical or the generalized (in the sense of Kellogg and Wiener) Green's function of D_∞ with pole at infinity. If, in particular, D_∞ is a simply connected domain, the author remarks that $P(z)$ is then single-valued and, setting $w=P(z)$, maps D_∞ conformally onto the circle $|w|>v(E)$ so that the points $z=\infty$ and $w=\infty$ correspond and w/z tends to 1 as $z\to\infty$.

W. Seidel (Notre Dame, Ind.)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

7155:

Ursell, H. D. Simultaneous linear recurrence relations with variable coefficients. Proc. Edinburgh Math. Soc. (2) 9 (1958), 183–206.

The author considers a system of difference equations

$$\sum_{1\leqq j\leqq k} A_{ij}u_j=z_i \ (1\leqq i\leqq n),$$

where each z_i is a given function on the set N of natural numbers, each u_j is an unknown function on N, and each A_{ij} is a given "polynomial operator", that is, an operator of the form $\sum a_\nu E^\nu$ where each a_ν is a function on N and E is the operator such that $(Ef)(n)=f(n+1)$. As in the classical theory of modules over a principal ideal ring, he introduces certain linear transformations to reduce the matrix (A_{ij}) to triangular or even diagonal form. Since the ring of functions on N has divisors of zero, there is difficulty with the Euclidean algorithm, and the reviewer was unable to see a way around. A footnote suggests a wider range of applicability, for example, to a system of linear differential equations. If the differential equations are taken over a differential field, or if the difference equations are taken over a difference field, his method will apply, being in fact a special case of the classical theory mentioned above [van der Waerden, *Moderne Algebra*, Springer, Berlin, 1930–31; § 106].

E. R. Kolchin (New York, N.Y.)

7156:

Kurepa, Svetozar. A cosine functional equation in n-dimensional vector space. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II. 13 (1958), 169–189. (Serbo-Croatian summary)

A continuous real function of the real variable t, if it is not identically zero or one and satisfies the functional equation

$$\phi(s+t)+\phi(s-t)=2\phi(s)\phi(t),$$

is of one of the forms $\phi(t)=\cos at$ or $\phi(t)=\cosh at$, where a is a constant. In this paper the same functional equation is investigated, on the assumption that the values $\phi(t)$ are linear transformations in a real n-dimensional vector space, and the variables s, t range over

the class of numbers of the form $l/2^k$, where l is an integer and k is a nonnegative integer. We cite two of the theorems. If $\phi(2^{-k})\to I$ (the identity) as $k\to+\infty$, and if $\phi(2^{-k})$ does not have 1 as eigenvalue when k is sufficiently large, then $\phi(t)=\cos tA$ for a suitable linear transformation A (the cosine defined by an infinite series). If now we assume $\phi(t)$ defined for all real t, $\phi(0)=I$, and $\phi(t)$ continuous and uniformly bounded, then with a suitable basis A can be taken to have a diagonal matrix representation.

A. E. Taylor (Los Angeles, Calif.)

SEQUENCES, SERIES, SUMMABILITY

See also 7201.

7157:

Climescu, Al. Critères d'existence pour la limite d'une suite à termes réels. Bul. Inst. Politehn. Iaşi (N.S.) 4(8) (1958), 17–22. (Romanian. Russian and French summaries)

The author proves two theorems concerning the convergence of series of real numbers. The first follows: Suppose ρ is a binary relation over the reals such that i) $a\rho b$ implies $(a+h)\rho(b+h)$ for all real h; ii) if $\{a_n\}$ is a sequence and $a_n\rho a_{n+1}$, $n=1, 2, \cdots$, then $\{a_n\}$ converges. Conclusion: ρ is monotone, that is $a\rho b\Rightarrow a\geqq b$ (or $a\leqq b$). The second theorem gives necessary and sufficient conditions for the convergence of $\{a_n\}$ in terms of the behavior of the double series $\{b_{ij}\}$ where $b_{ij}=(a_i+a_j)/2$.

E. R. Lorch (New York, N.Y.)

7158:

Lorentz, G. G.; and Zeller, K. Series rearrangements and analytic sets. Acta Math. 100 (1958), 149–169.

Let A be an infinite matrix with real elements a_{nm}, and let $\sum u_m$ be a series of real terms. The relations $v_n=\sum_{m=1}^{\infty} a_{nm}u_m$, $n=1, 2, \cdots$, define a series-to-series transformation, denoted by $v=Au$, and the series $v=\sum v_n$ is called the A-transform of the series $u=\sum u_m$. A is called regular if it transforms every convergent series into a convergent series with the same sum. The A-rearrangement set of $\sum u_m$ is defined as the set of all numbers v for which there exists a rearrangement of $\sum u_m$ which is A-summable to v. (For instance, if I denotes the identity matrix, the I-rearrangement set of $\sum u_m$ is either the null set, or a set consisting of one single number, or the set of all real numbers.)

The two principal results of the paper are the following. (1) For any matrix A, regular or not, and any series $\sum u_m$, the A-rearrangement set of $\sum u_m$ is an analytic set. [An analytic set is a Suslin set generated by the system of closed sets. See, for instance, F. Hausdorff, *Mengenlehre*, 3rd edition, Gruyter, Berlin-Leipzig, 1935; §§ 19 and 32.] (2) If T is an analytic set of real numbers then there exists a regular matrix A and a series $\sum u_m$, such that T is the A-rearrangement set of $\sum u_m$. {In fact, the authors use in their proof of (2) one single series, namely, $\exp 1!+0+\exp 2!+0+\exp 3!+0+\cdots$, but their proof would equally apply to any series of the form $u_1+0+u_2+0+u_3+0+\cdots$ with positive non-decreasing u_n and $u_{n+1}/u_n\to\infty$.}

Generalizations of the results (1) and (2) are given in two different directions. In the first place, both results remain true if one considers matrices and series with complex terms and analytic sets of complex numbers. Secondly, (2) can be extended to sequence-to-series and sequence-to-sequence methods of summability.

F. Herzog (East Lansing, Mich.)

7159:

Tschobanow, W.; and Paskalew, G. Zu linearen Limitierungsverfahren. Studia Math. 17 (1958), 141–149.

Die Verfasser betrachten normale Matrizen A, bei denen die Zeilensummen einem Grenzwert zustreben, und untersuchen Fragen der Translation (Indexverschiebung) bei den zugehörigen Limitierungsverfahren A. Linkstranslativität von A bedeutet, daß mit jeder Folge $(s_0, s_1, \cdots)$ auch die Folge $(s_1, s_2, \cdots)$ von A limitiert wird. Wegen der Zeilensummenbedingung läuft es auf dasselbe hinaus zu fordern, daß mit jeder Reihe $a_0+a_1+\cdots$ auch die Reihe $a_1+a_2+\cdots$ von A summiert wird. Ergänzt man die Matrix $(a_{m-1,n-1})$ in naheliegender Weise zu einer normalen Matrix A', so bedeutet Linkstranslativität von A, daß A' jede A-summierbare Reihe summiert, d.h. daß $A'A^{-1}$ konvergenztreu ist. Letzteres wird in Hilfssatz 2 mittels Bedingungen vom Toeplitz-Typ formuliert (Spezialfall eines Satzes von Mazur), wobei auch die A'-Summe durch die A-Summe ausgedrückt wird. Andererseits ist klar, daß Linkstranslativität äquivalent mit folgender Eigenschaft ist: Die Folge (a_{n+1}) ist A-limitierbar, wenn A die Reihe $\sum a_n$ summiert. Man schließt weiter, daß aus Linkstranslativität mit Verträglichkeit $A\text{-}\lim(a_{n+1}+\cdots+a_{n+p})=0$ für jedes feste p folgt (Satz 2). Die Sätze 3 und 4 bringen hinreichende Translativitätsbedingungen, in denen im wesentlichen gefordert wird, daß die m-te Spalte von A^{-1} ein Vielfaches der um 1 verschobenen $(m-1)$-ten Spalte ist. Die Verfasser wenden ihre Ergebnisse auf Nörlundverfahren an.

K. Zeller (Tübingen)

7160:

Erwe, Friedhelm. Zur Limitierung der beschränkten Folgen. Arch. Math. 9 (1958), 197–201.

Let $\mathfrak{B}$ denote the space of bounded real sequences, and $\mathfrak{F}$ the class of linear functionals L on $\mathfrak{B}$ that satisfy the inequality $L(\mathfrak{a})\leqq\limsup\mathfrak{a}$ for all sequences $\mathfrak{a}$ in $\mathfrak{B}$. Let $\mathfrak{d}$ be a proper dual sequence, that is to say, a sequence $\{d_n\}$ with $d_n=0$ or $d_n=1$ for each n, and with $d_n=1$ for infinitely many n. Given a sequence $\mathfrak{a}$, $\mathfrak{d}\times\mathfrak{a}$ denotes the sequence obtained by replacing (for each n) the nth 1 in $\mathfrak{d}$ by the nth term in $\mathfrak{a}$. It is proved that if also $d_n=0$ for infinitely many n, and $0\leqq t\leqq 1$, then there exist elements L of $\mathfrak{F}$ such that $L(\mathfrak{d}\times\mathfrak{a})=tL(\mathfrak{a})$ for all $\mathfrak{a}$ in $\mathfrak{B}$. It is also proved that there exist elements L of $\mathfrak{F}$ such that $L(\mathfrak{k}_1\times\mathfrak{a})=L(\mathfrak{a})$ and $L(\mathfrak{e}'\times\mathfrak{a})=\frac{1}{2}L(\mathfrak{a})$ for all $\mathfrak{a}$ in $\mathfrak{B}$, where $\mathfrak{k}_1=(0, 1, 1, \cdots)$ and $\mathfrak{e}'=(1, 0, 1, 0, \cdots)$. The proofs depend on the following lemma. Given a set $\mathfrak{M}$ of linear mappings of $\mathfrak{B}$ into itself, there exist functionals L in $\mathfrak{F}$, with $L(\mathfrak{H}(\mathfrak{a}))=0$ for all $\mathfrak{a}$ in $\mathfrak{B}$ and all $\mathfrak{H}$ in $\mathfrak{M}$, if and only if $\limsup\mathfrak{b}\geqq 0$ for all $\mathfrak{b}$ of the form $\mathfrak{b}=\mathfrak{H}_1(\mathfrak{a}_1)+\cdots+\mathfrak{H}_n(\mathfrak{a}_n)$, with $\mathfrak{H}_1, \cdots, \mathfrak{H}_n$ in $\mathfrak{M}$ and $\mathfrak{a}_1, \cdots, \mathfrak{a}_n$ in $\mathfrak{B}$.

F. F. Bonsall (Newcastle-upon-Tyne)

7161:

Meyer-König, W.; und Zeller, K. Zum Vergleich der Verfahren von Cesàro und Abel. Arch. Math. 9 (1958), 191–196.

Let C_α denote the Cesàro method of summation of order α. It is known that if $0<\alpha<\beta$, then every sequence summable C_α is also summable C_β, while for bounded sequences the converse holds: C_β summability implies C_α summability. The authors prove the following theorem. Let the sequence $w_0\leqq w_1\leqq\cdots\to\infty$, and the two numbers α, β $(0<\alpha<\beta)$ be given. Then there exists a sequence

$\{s_k\}$ with $s_k=o(w_k)$ that is summable C_β but not C_α. From this it follows that if $a>0$ is given, then the intersection of the methods C_α $(\alpha>a)$ is not equivalent to any matrix method [see Zeller, Math. Z. **59** (1953), 271–277; MR **15**, 618].

If $\alpha>0$ then every sequence summable C_α is also summable A (Abel's method). The converse is false in general, but the authors prove that if $\{s_k\}$ is A-summable to zero and if $\limsup |(1-\rho)\sum_m^\infty S_k\rho^k|=0$, where the limit is on m and the sup is on $0<\rho\leq 1$, then $\{s_k\}$ is C_1-summable to zero. *A. L. Shields* (Ann Arbor, Mich.)

7162:

★**Meyer-König, W.; und Zeller, K. Funktionalanalytische Behandlung des Taylorschen Summierungsverfahrens.** Colloque sur la théorie des suites, tenu à Bruxelles du 18 au 20 decembre 1957, pp. 32–53. Centre Belge de Recherches Mathématiques. Librairie Gauthier-Villars, Paris; Établissements Ceuterick, Louvain; 1958. 167 pp. 220 fr. belges.

The authors discuss applications of functional analysis to the Taylor transform T_α $(0<\alpha<1)$:

$$v_m=(1-\alpha)^m \sum_{k=m}^{\infty} \binom{k}{m} \alpha^{k-m} u_k \quad (m=0, 1, \cdots)$$

of the series $\sum u_k$ into the series $\sum v_m$. In the space $\mathfrak{T}_\alpha$ of all T_α-summable series a natural FK-topology is given by a sequence of semi-norms. $\mathfrak{T}_\alpha$ is not a Banach BK space; as a corollary, $\mathfrak{T}_\alpha$ is not the summability field of any row-finite method. The authors also determine the perfect part of $\mathfrak{T}_\alpha$ (the set of all points approximable in the topology of $\mathfrak{T}_\alpha$ by finite series $\sum u_k$): this is the set of regularly T_α-summable series $\sum u_k$ for which $\sum u_k z^k$ is regular for $z=\alpha$. Also the gap-perfect part of $\mathfrak{T}_\alpha$ [see Meyer-König and Zeller, Math. Z. **66** (1956), 203–224; MR **18**, 733] is the same set for $0<\alpha\leq\frac{1}{2}$. This leads to a Tauberian theorem for regularly T_α-summable series with gaps between k_l, $k_{l+1}-k_l\geq \text{const}\sqrt{k_l}$. *G. G. Lorentz* (Syracuse, N.Y.)

7163:

★**Vermes, P. The transpose of a summability matrix.** Colloque sur la théorie des suites, tenu à Bruxelles du 18 au 20 decembre 1957, pp. 60–86. Centre Belge de Recherches Mathématiques. Librairie Gauthier-Villars, Paris; Établissements Ceuterick, Louvain; 1958. 167 pp. 220 fr. belges.

Generalizing an observation of Ramanujan [J. London Math. Soc. **32** (1957), 27–32; MR **18**, 732] about Hausdorff summation methods, the author shows that the transpose of a regular sequence to sequence matrix A is a regular series to series matrix if and only if the matrix A has row sums equal to one and is absolutely regular. There are several related theorems. The author investigates the invariance of the properties in question under the operations of multiplication, convolution and weighted sum of matrices, and gives applications to special methods. *G. G. Lorentz* (Syracuse, N.Y.)

7164:

Vermes, P. Summability of power series at unbounded sets of isolated points. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 830–838.

It was first shown in 1939 by P. Dienes and the reviewer [Proc. London Math. Soc. (2) **45** (1939), 45–63; see pp. 53–55 {an incorrect reference for this is given by the author at the end of the paper}] that a regular summability method can be efficient at an isolated arbitrary point α outside the circle of convergence, for the series $\sum z^n$; they also constructed another T-matrix which has this property at the isolated points $z=\alpha^{1/m}$, where m is any positive integer. In 1947, the author [Proc. Edinburgh Math. Soc. (3) **8** (1947), 1–13; MR **9**, 234; see pp. 11–13] considered series-to-sequence transformations, and constructed a γ-matrix efficient for the binomial series $\sum_{k=0}^\infty \binom{k+p-1}{k} z^k$ when $p=1, 2, \cdots, q$ at the isolated points $z=\alpha_1, \cdots, \alpha_m$, and inefficient when $p>q$; the complex numbers $\alpha_1, \cdots, \alpha_m$, with moduli >1 and the integer q are involved in the construction of the matrix. It was seen to be impossible to extend this method to an infinity of isolated points.

In 1952 S. E. Tolba [Nederl. Akad. Wetensch. Proc. Ser. A **55** (1952), 380–387; MR **14**, 369] extended these results to a class of power series $z^q\sum_{k=0}^\infty c_k z^{pk}$, where p, q are positive integers and $|c_k/c_{k+1}|\to 1$ as $k\to\infty$. In 1951 the author constructed a δ-matrix (conservative series-to-series transformation matrix) efficient for $\sum z^n$ at a countable infinity of points outside $|z|=1$ [Acta Sci. Math. Szeged **14** (1951), 23–38; MR **13**, 27; pp. 36–37], and in 1957 [#7163 above] he applied the same matrix to sequences; if a certain condition is satisfied, the matrix is a T-matrix (i.e., regular).

On a suggestion of the reviewer, D. C. Russell [Ph. D. Thesis, London (1958); the results will be published very soon] constructed T-matrices efficient for $\sum z^n$ at all finite points of isolated continuous curves outside $|z|=1$.

In the present paper the author, following a suggestion of M. R. Mehdi, constructs T-matrices efficient for every binomial series at a countable infinity of isolated points outside the circle of convergence; he also shows that some of these matrices have the same property for the Taylor series of a class of meromorphic functions. *R. G. Cooke* (London)

7165:

Fekete, M. New methods of summability. J. London Math. Soc. **33** (1958), 466–470.

In 1916 the author showed that analytic continuation of a power series can be represented as a matrix-transformation of the partial sums by the upper triangular matrix $t_{nk}=\binom{k}{n} r^{k-n}(1-r)^{n+1}$. This appeared in a Hungarian textbook by Beke [*Differentiál és Integrál számítás*, vol. 2, 1916] an English summary of which is to be found in a paper of Vermes [Amer. J. Math. **71** (1949), 541–562; MR **10**, 699]. This method of summability has subsequently been called the Taylor method. In the present note (a summary of a lecture given by the author in 1954 as compiled by P. Vermes) two new methods of summability are introduced. The first is a lower triangular matrix called Taylor-Nörlund. Let P_k be a sequence of positive numbers such that $p_k/P_k\to 0$ where $P_k=p_0+\cdots+p_k$. This gives rise to a regular Nörlund matrix $p_{n,k}=p_{n-k}/P_n$. Assume also that $\sum p_k z^k$ has unit radius of convergence. Then with $q_k=(1-r)^k\sum_{m=k}^\infty \binom{m}{k} p_m r^{m-k}$ it can be shown that $q_{n,k}=q_{n-k}/Q_n$, $Q_n=q_0+\cdots+q_n$, is a regular Nörlund transform. Set $T=(t_{nk})$ and write $t(z)=TS(z)$ for a sequence $\{S_k(z)\}$. Then the Taylor-Nörlund transform of the sequence $S_k(z)$ is given by

$$h_n(z)=\sum_{k=0}^{n} \frac{t_k(z)q_{n-k}}{Q_n}.$$

In a similar manner an upper triangular transform called Nörlund-Taylor is introduced. *V. F. Cowling* (Lexington, Ky.)

7166:

Tatchell, J. B. A note on matrix summability of unbounded sequences. J. London Math. Soc. 34 (1959), 27–36.

The author considers matrix summability methods A with the property (*): Every column of A is a convergent sequence and $\lim_\nu \sup_n |a_{n\nu}|=0$. (This condition is satisfied by most of the common summability methods.) Given any pair of such methods, there is an unbounded sequence summable by both methods. On the other hand no unbounded sequence is summable by all such methods. From these lemmas follows the main result, theorem 1: When a matrix method satisfies (*), its summability space contains an enumerable set of unbounded sequences which are linearly independent of bounded sequences. — The theorem can be generalized: Weakening of condition (*), imposing of order restrictions on the unbounded sequences. The author uses FK-spaces. {An alternative way would be to apply "factor sequences", which seem to yield even stronger results [compare, e.g., the reviewer's note in Math. Z. 56 (1952), 134–151; MR 14, 158]}. The author gives several examples concerning Nörlund methods. *K. Zeller* (Tübingen)

7167:

Ramanujan, M. S. On Hausdorff and quasi-Hausdorff methods of summability. Quart. J. Math. Oxford Ser. (2) 8 (1957), 197–213.

Let (H, μ_n) and (H^*, μ_n) denote a Hausdorff matrix and its transpose, respectively. The author shows that the sequence-to-sequence transformation represented by (H^*, μ_n) is absolutely conservative or absolutely regular if and only if it is conservative or regular, respectively. The same is true of the series-to-series transformation represented by (H, μ_n).

In the space of bounded sequences, a regular sequence-to-sequence transformation (H, μ_n) or (H^*, μ_n) includes the Borel exponential method if and only if $\mu_n \to 0$.

The author also introduces the matrices $S^* \equiv (S^*, \mu) = (s_{nk}^*)$, where

$$s_{nk}^* = \binom{n+k}{k} \Delta^k \mu_{n+1} \quad (n, k=0, 1, \cdots).$$

The matrix S^* is regular if and only if μ_n is a moment sequence with $\mu_n \to 0$ and $\int_{0+}^1 d\chi(t)=1$. The method $S(\alpha)$ of Meyer-König is a special case of S^*.

G. Piranian (Ann Arbor, Mich.)

7168:

Kuttner, B. Some remarks on quasi-Hausdorff transformations. Quart. J. Math. Oxford Ser. (2) 8 (1957), 272–278.

(Notation as above.) The author shows that the inclusion relation $(H, \mu_n) \supset (H^*, \mu_{n+1})$ and its reverse do not generally hold. The relation fails if H is a regular Euler matrix (here (H^*, μ_{n+1}) is a Meyer-König matrix); the reverse fails if H is a Cesàro matrix of order greater than 2.

G. Piranian (Ann Arbor, Mich.)

7169:

Kuttner, B. Note on a paper by M. S. Ramanujan on quasi-Hausdorff transformations. J. Indian Math. Soc. (N.S.) 21 (1957), 97–104.

The author gives a new proof (free of the theory of moment sequences) of a theorem of M. S. Ramanujan [same J. 17 (1953), 47–53; MR 15, 118].

G. Piranian (Ann Arbor, Mich.)

7170:

Polniakowski, Z. Polynomial Hausdorff transformations. I. Mercerian theorems. Ann. Polon. Math. 5 (1958), 1–24.

The author obtains conditions for regularity and Mercerian theorems for certain special classes of Hausdorff transformations. His work is related to that of H. R. Pitt [Proc. Cambridge Philos. Soc. 34 (1938), 510–520], but is not as general, and the methods used are accordingly much more elementary. The author first considers the Hausdorff transformation

$$s_p \to t_p = \sum_{n=0}^{p} (-1)^n \binom{p}{n} \mu_n \Delta^n s_0,$$

with $\mu_n = W_1(n)/W(n)$. Here W is a polynomial of degree k and W_1 a polynomial of degree $\leqq k$, $W(0)=W_1(0)=1$. He shows that this transformation is regular if and only if all zeros of W have negative real part. He next proves the following Mercerian theorem for the case $W_1=1$: in order that for every sequence $\{s_n\}$ the hypothesis $\alpha s_n + (1-\alpha)t_n \to s$ (with $\alpha \neq 0$) implies $s_n \to s$ it is necessary and sufficient that all zeros of $W-1+1/\alpha$ have negative real part. More general Mercerian theorems are proved also, and analogous results are obtained for the related transformations

$$t(x) = \sum_{n=0}^{\infty} \mu_n \binom{x}{n} \Delta^n f(0), \quad t(x) = \sum_{n=0}^{\infty} \mu_n (x^n/n!) f^{(n)}(0).$$

J. Korevaar (Madison, Wis.)

7171:

Rhoades, B. E. Some structural properties of Hausdorff matrices. Bull. Amer. Math. Soc. 65 (1959), 9–11.

The author gives twelve theorems, without proofs, from which he concludes that it is impossible to construct by the method of Zeller [Arch. Math. 4 (1953), 1–5; MR 14, 866] a Hausdorff matrix with finite norm whose convergence domain is the set of sequences $c \oplus x = \{y+x | y \in c\}$, for some unbounded sequence x, where c is the space of convergent sequences. *D. Moskovitz* (Pittsburgh, Pa.)

7172:

Pitt, H. R. A general Tauberian theorem related to the elementary proof of the prime number theorem. Proc. London Math. Soc. (3) 8 (1958), 569–588.

The author proves a general Tauberian theorem, which can be used in the elementary proof of the prime number theorem as given by Selberg [Ann. of Math. (2) 50 (1949), 305–313; MR 10, 595]. Transformations of the form

$$(*) \qquad g(x) = s(x) + \frac{1}{x} \int_0^x s(x-y)\,dk(y) \quad (x>0)$$

are abelian, in the sense that $s(x) \to A$ implies that $g(x) \to 2A$, provided that $k(0)=0$, $k(x)$ increases for $x \geqq 0$, and $k(x) \sim x$ as $x \sim \infty$. Furthermore, it is assumed that $\limsup [k(x)-k(x-\delta)] = \eta(\delta) < \infty$ for every positive constant δ. The object of this paper is to find conditions under which the Tauberian theorem that $g(x) \to 2A$ implies that $s(x) \to A$ can be asserted. It is found that the appropriate Tauberian condition is that $s(x)$ should satisfy the Schmidt slowly decreasing condition that $\liminf [s(x')-s(x)] \geqq 0$ as $x \to \infty$, $x' \geqq x$, $x'-x \to 0$ [Math. Z. 22 (1924), 89–152; see pp. 127–142].

It is to be noted that the Tauberian theorem fails (with $s(x) = \cos(\pi x/\lambda)$) if $k(x)$ is a step function with positive increments 2λ at the points $(2j+1)\lambda$ $(j=0, 1, 2, \cdots)$, so that some further condition is required on $k(x)$ to avoid this situation.

If we put $k(x)=\sum \log p/p$ (p prime, $\log p\leqq x$) and $s(x)=e^{-x}\vartheta(e^x)-1$, $\vartheta(u)=\sum \log p$ ($p\leqq u$), (*) becomes the kernel of Selberg's proof, and this deduction of the prime number theorem is essentially the same as Selberg's. [For a general account, see § 6.3 of the author's *Tauberian theorems*, Oxford Univ. Press, London, 1958.]

S. Ikehara (Tokyo)

7173:

★**Karamata, J. Sur les procédés de sommation intervenant dans la théorie des nombres.** Colloque sur la théorie des suites, tenu à Bruxelles du 18 au 20 décembre 1957, pp. 12–31. Centre Belge de Recherches Mathématiques. Librairie Gauthier-Villars, Paris; Établissements Ceuterick, Louvain; 1958. 167 pp. 220 fr. belges.

The author discusses various Tauberian theorems which are of importance in prime number theory. Put

$$(1)\qquad G(x)=\sum_{n\leq x} f(n)[x/n]=Ax\log x+Bx+P(x).$$

If we only assume $P(x)=o(x)$ then if $f(n)\geqq -M$ ($1\leqq n<\infty$) it is shown that

$$(2)\qquad \sum_{n\leq x} f(n)=Ax+o(x).$$

The proof utilises $\sum_{n=1}^{x} \mu(n)=O(x/(\log x)^2)$ ($\mu(n)$ is the Möbius symbol). If in (1) we assume $\rho(x)=o(g(x))$, $g(x)$ monotone increasing, where $\int_1^\infty g(x)/x^2<\infty$, then (2) follows without the condition $f(n)>-M$.

If $\sum_{n\leq x} f(n)[x/n]=O(x)$, then $\int_1^x F(t)/t=O(x)$, where $F(n)=\sum_{k=1}^{n} f(k)$. All these results are connected with the prime number theorem.

Several further results are discussed and there is an extensive bibliography. *P. Erdős* (Birmingham)

APPROXIMATIONS AND EXPANSIONS

See also 7121, 7381.

7174:

Schoenberg, I. J. Spline functions, convex curves and mechanical quadrature. Bull. Amer. Math. Soc. **64** (1958), 352–357.

By a spline function of degree $n-1$ is meant a function of the form

$$S_{n-1,k}(x)=P_{n-1}(x)+\sum_{\nu=1}^{k} C_\nu(x-\xi_\nu)_+^{\,n-1},$$

where $P_{n-1}(x)$ is a polynomial of degree $\leqq n-1$ and $x_+^{\,n-1}=x^{n-1}$ for $x\geqq 0$ and 0 if $x<0$. In this research announcement a fundamental theorem of algebra is given for spline functions and applications are indicated to mechanical quadrature formulas of Gauss and Radau type. The determination of the knots (ξ_ν) of a spline function with given zeros is made to depend upon a refinement of a theorem of Carathéodory on convex hulls.

P. J. Davis (Washington, D.C.)

7175:

Hsu, L. C.; and Lin, L. W. Two new methods for the approximate calculation of multiple integrals. Acta Math. Acad. Sci. Hungar. 9 (1958), 279–290.

The first method involves the approximation of an mth-fold integral of a function f periodic of period 2π in each of its m variables. The approximation is made by the integral mean of the function φ whose value is $\varphi(t)=f(R^{q-1}t, R^{q-2}t, \cdots, R^{q-m}t)$. The order of approximation is given. In the second method the requirement of periodicity is dropped and more elaborate approximants are used.

P. Civin (Eugene, Ore.)

7176:

Balázs, J. Bemerkungen zur Hermite-Fejérschen Interpolationstheorie. Acta Math. Acad. Sci. Hungar. **9** (1958), 363–377.

Let $f(x)$ be defined on I: $[-1, 1]$ and let $1\geqq\xi_{1n}>\xi_{2n}>\cdots>\xi_{nn}\geqq -1$ ($n=1, 2, \cdots$) be a system of points lying in that interval. The Hermite formula

$$H_n(f;x)=\sum_{\nu=1}^{n} f(\xi_{\nu n})h_{\nu n}(x)+\sum_{\nu=1}^{n} \alpha_{\nu n}\mathfrak{h}_{\nu n}(x),$$

with

$$h_{\nu n}(x)=\{1-\frac{w_n''(\xi_{\nu n})}{w_n'(\xi_{\nu n})}(x-\xi_{\nu n})\}\{l_{\nu n}(x)\}^2,$$

$$\mathfrak{h}_{\nu n}(x)=(x-\xi_{\nu n})\{l_{\nu n}(x)\}^2,$$

$$w_n(x)=c(x-\xi_{1n})\cdots(x-\xi_{nn})\ (c\neq 0),$$

$$l_{\nu n}(x)=\frac{w_n(x)}{w_n'(\xi_{\nu n})(x-\xi_{\nu n})},\ \alpha_{\nu n}\text{ arbitrary},$$

yields a polynomial of degree $\leqq 2n-1$ for which $H_n(f;\xi_{\nu n})=f(\xi_{\nu n})$ and $H_n'(f;\xi_{\nu n})=\alpha_{\nu n}$. L. Fejér initiated the study of the convergence of H_n to f when the ξ's are the zeros of the Jacobi polynomials. G. Freud [Acta Math. Acad. Sci. Hungar. **5** (1954), 109–128; MR **16**, 694] recently studied this convergence when the ξ's are the zeros of orthogonal polynomials corresponding to a general non-negative weight. P. Turán raised some questions as to what happens when the weighting function vanishes in the interval, and in answer to these questions the author has shown the following. Let $\xi_{1n}, \xi_{2n}, \cdots, \xi_{nn}$ be the zeros of the polynomials $p_n(x)$ which are orthogonal on I with respect to the weight $p(x)=(1-x^2)^{-\frac{1}{2}}|x|^{2\mu+1}$ ($-\frac{1}{2}<\mu<0$). Let $f(x)$ be continuous on I and $|\alpha_{\nu n}|<c_1n^\delta$, $|\mu|>\delta\geqq 0$. Then the corresponding sequence of polynomials H_n converges to f uniformly on I. Moreover, the "conjugate points" $x_{\nu n}=\xi_{\nu n}+p_n'(\xi_{\nu n})/p_n''(\xi_{\nu n})$ are everywhere dense in I. The orthogonal polynomials corresponding to the weight functions $(1-x^2)^p|x|^q$ have been determined by K. V. Laščenov [Leningrad. Gos. Ped. Inst. Uč. Zap. **89** (1953), 167–189; MR **17**, 730].

P. J. Davis (Washington, D.C.)

7177:

Ribarič, M.; and Suhadolc, A. On the completeness of orthogonal polynomials in infinite intervals. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II. **13** (1958), 165–168. (Serbo-Croatian summary)

The authors discuss the closure problem for $\{x^nq(x)\}_{n=0}^{\infty}$ in $L^2[0, \infty]$ and $L^2[-\infty, \infty]$ by recasting this in terms of the determinacy of a Stieltjes or Hamburger moment problem, and applying known criteria. Application to $q(x)=|x|^\alpha \exp(-c|x|^p)$. *R. C. Buck* (Stanford, Calif.)

7178:

Alexits, Georges. Une contribution à la théorie constructive des fonctions. Acta Sci. Math. Szeged **19** (1958), 149–157.

Let $\{p_n(x)\}$ be a system of polynomials, orthonormal with respect to a weight function $\rho(x)\geqq 0$ on $\langle a, b\rangle$. In generalisation of a familiar theorem of S. Bernstein the following result is obtained. Suppose that, on a subinterval $\langle c, d\rangle$, one has $(n+1)^{-1}\sum_0^n p_k^2(x)\leqq C_1$ and

$0 \leq \rho(x) \leq C_2$. Let $f \in L_\rho^2\langle a, b\rangle$ and $f \in \text{Lip}\,\alpha$ on $\langle c, d\rangle$, where $0<\alpha<\frac{1}{2}$. Then, for each $\delta>0$

$$(*) \quad \frac{1}{n+1}\sum_0^n |s_k(x)-f(x)| \leq \frac{C(\delta)}{n^\alpha} \text{ on } \langle c+\delta, d-\delta\rangle.$$

Conversely, if (*) holds, then $f \in \text{Lip}\,\alpha$ on $\langle c, d\rangle$, provided that $0<m\leq\rho(x)\leq M$ there. These results are further extended to certain other orthonormal systems.

W. W. Rogosinski (Newcastle-upon-Tyne)

7179:

Chen, Kien-Kwong. Approximation by Cesàro combination of Faber's polynomials on the continuum having fairly smooth boundary. Rev. Math. Pures Appl. 1 (1956), no. 3, 113–146.

D is the interior of a rectifiable Jordan curve Γ with continuous tangent in the complex z-plane. Let s be the length of arc and $\vartheta(s)$ be the tangential angle with the x-axis. If $\omega(t)=\max_{|s_1-s_2|\leq t}|\vartheta(s_1)-\vartheta(s_2)|$ is the modulus of continuity of $\vartheta(s)$, it is further assumed that $t^{-1}\log(t^{-1}\omega(t))$ be integrable on $0\leq t\leq 1$. Next, let $f(z)$ be regular in D, continuous on Γ, and let it be the uniform derivative of an $F(z)$ bounded in D. If the $\Phi_k(z)$ are the Faber polynomials of Γ, let $f(z)\sim\sum_0^\infty a_k\Phi_k(z)$, and denote, for $\gamma>0$, the nth Cesàro sum of order γ of this series by $\sigma_n{}^\gamma(z)$. Amongst others the following theorem is proved: Let $\Omega(t)$ be the modulus of continuity of $f(z)$ on Γ and let $\alpha=\limsup_{t\to 0}(t\Omega'(t)/\Omega(t))$. If $\alpha<1$, then

$$|\sigma_n^\gamma(z)-f(z)| \leq \frac{C_\gamma}{1-\alpha}\Omega\left(\frac{1}{n+1}\right);$$

if $\alpha=1$, then

$$|\sigma_n{}^\gamma(z)-f(z)| \leq C_\gamma\Omega\left(\frac{1}{n+1}\right)\log\frac{1}{n+1},$$

for all $z \in D\cup\Gamma$. If Γ is $|z|=1$, when one has power series, the following result is also of interest: If $\alpha<1$, then $|f'(z)|\leq C\Omega(1-|z|)/(1-|z|)$; if $\alpha=1$, then

$$|f'(z)|\leq C\Omega(1-|z|)/(1-|z|)\log(1-|z|)^{-1}$$

for $|z|<1$. Conversely, if for an increasing $\lambda(t)$, $|f'(z)| \leq B\lambda(1-|z|)/(1-|z|)$, then $\Omega(t)\leq 5B\lambda(t)$. There are similar results for Fourier series of continuous functions.

W. W. Rogosinski (Newcastle-upon-Tyne)

7180:

Pták, Vlastimil. On approximation of continuous functions in the metric $\int_a^b |x(t)|dt$. Czechoslovak Math. J. 8(83) (1958), 267–273; supplement, 464.

Let E be an n-dimensional subspace of $L_1[a, b]$ such that each y in E is continuous on $[a, b]$. The author gives an extremely short proof of the following theorem: Each continuous x in $L_1[a, b]$ has a unique nearest point in E (with respect to the L_1 norm) provided each y in E, $y\neq 0$, has at most $n-1$ zeros in the interior of $[a, b]$. {Part of the proof is contained in the supplement.} [Another proof of this theorem appears in N. I. Ahiezer and M. G. Kreĭn, *On some problems in the theory of moments*, Gosudarstv. Nauč.-Tehn. Izdat. Ukraïn., Harkov, 1938; Ch. 4; and a slightly less general version in N. I. Ahiezer, *Theory of approximation*, Ungar Publ. Co., New York, 1956; MR 20 #1872; pp. 76–81.] Some related results are obtained.

R. R. Phelps (Princeton, N.J.)

FOURIER ANALYSIS

See also 7103, 7104.

7181:

Pólya, G.; and Schoenberg, I. J. Remarks on de la Vallée Poussin means and convex conformal maps of the circle. Pacific J. Math. 8 (1958), 295–334.

Let $v(a)$ designate the number of variations of signs in the terms of the finite sequence $a_1, a_2, \cdots, a_n$ of real numbers. Let $v_c(a)$ designate the number of cyclic variations of our sequence, i.e., we set $v_c=0$ if all $a_\nu=0$; and if $a_i\neq 0$, we set

$$v_c(a)=v(a_i, a_{i+1}, \cdots, a_n, a_1, a_2, \cdots, a_{i-1}, a_i).$$

If f is a real-valued function of period 2π, then the number $v_c(f)$ of cyclic variations of sign of $f(t)$ is defined by $v_c(f)=\sup v_c(f(t_\nu))$ the supremum being taken for all finite sequences $t_1<t_2<\cdots<t_n<t_1+2\pi$. Now consider a non-negative weight-function (kernel) $\Omega(t)$ of period 2π, of bounded variation and normalized by the conditions

$$(2\pi)^{-1}\int_0^{2\pi}\Omega(t)dt=1, \quad \Omega(t)=\tfrac{1}{2}(\Omega(t+0)+\Omega(t-0)).$$

Let f be any real-valued, integrable function of period 2π and form its convolution transform

$$g(t)=(2\pi)^{-1}\int_0^{2\pi}\Omega(t-\tau)f(\tau)d\tau.$$

Then the authors call this transformation "variation diminishing" if $v_c(g)\leq v_c(f)$ for each f, and $\Omega(t)$ is called a "variation diminishing kernel".

The authors now consider the kernels of C. J. de la Vallée Poussin [Acad. Roy. Belg. Bull. Cl. Sci. 3 (1908), 193–254]

$$\omega_n(t)=\frac{(n!)^2}{(2n)!}\left(2\cos\frac{t}{2}\right)^{2n} \quad (n=1, 2, \cdots).$$

For $\Omega(t)=\omega_n(t)$, they denote $g(t)$ by $V_n(t)$ and call these transformations "de la Vallée Poussin means" or "V-means" of f. By $Z_c(V_n)$ designate the number of real zeros of $V_n(t)$ within a period, including multiplicities. Then the main result of the authors is the theorem 1: The inequalities $v_c(V_n)\leq Z_c(V_n)\leq v_c(f)$ hold for any integrable f. For this theorem two proofs are given, one using a theorem of J. J. Sylvester, the other one using the method of one of its proofs.

In Part II the authors discuss applications of the variation diminishing property of V-means. Previous results of C. Sturm, A. Hurwitz, and G. Pólya and N. Wiener are obtained as corollaries of theorem 1. The graphic behavior of V-means is then studied under rather weak assumptions, namely that either the classical Dirichlet conditions are satisfied or a decomposition into a finite number of consecutive open arcs is possible in each of which f is continuous and convex, or concave, or linear. Furthermore, considering a result of L. Fejér [Z. Angew. Math. Mech. 13 (1933), 80–88] on the third Cesàro means, the authors prove the following analogue for V-means: If f is an odd periodic function which is positive and concave for $0<t<\pi$, then $0\leq V_n(t)\leq f(t)$ if $0<t<\pi$ $(n\geq 1)$; moreover, $V_n(t)$ is also concave in $0<t<\pi$.

Then the authors study complex-valued functions. Let K denote the class of the "Schlicht" power series $\sum_{\nu=1}^\infty a_\nu z^\nu$

which map $|z|<1$ onto some convex domain. Let $f(z)=\sum_{\nu=1}^{\infty} c_\nu z^\nu$ $(c_1=1)$ and call

$$\mathbf{V}_n(z)=\frac{1}{\binom{2n}{n}}\sum_{\nu=1}^{n}\binom{2n}{n+\nu}c_\nu z^\nu$$

its de la Vallée Poussin mean or **V**-mean. The authors prove the following theorem 2: For $f(z)\in K$ it is necessary and sufficient that $\mathbf{V}_n(z)\in K$ for $n=1, 2, \cdots$.

Finally, two appendices are added. In appendix I it is shown that the Bernstein polynomials have the variation diminishing property and the graphic behavior of these polynomials is discussed. Appendix II states a conjecture on power series mapping a circle onto a convex domain, namely: If both series

$$a_1z+a_2z^2+a_3z^3+\cdots,$$
$$b_1z+b_2z^2+b_3z^3+\cdots$$

belong to K, then also

$$a_1b_1z+a_2b_2z^2+a_3b_3z^3+\cdots$$

belongs to K. Some particular cases and some consequences of this conjecture are verified by the authors.

A. Rosenthal (Lafayette, Ind.)

7182:

Buchwalter, Henri. Saturation de certains procédés de sommation. C. R. Acad. Sci. Paris **248** (1959), 909–912.

The summation process Γ defined by

$$\Gamma_w(S)=\sum_{k=0}^{\infty} \nu_{w,k}u_k,$$

where $S=\sum_0^\infty u_k$, and $\nu_{w,k}$ are real constants satisfying the Toeplitz conditions, is applied to the summation of Fourier series in the space C and L^p for $p\geq 1$. For $f\in L^p$ or C, let $f\sim\sum_0^\infty A_k(x)$, with $A_k(x)=a_k\cos kx+b_k\sin kx$, and let $\sum_1^\infty \bar{A}_k(x)$ denote the conjugate series of the Fourier series for f. Being given a sequence λ_k and a function $\phi(w)$ such that

(1) $$0\leq\lambda_0<\lambda_k<\lambda_{k+1}\to\infty;\ \lambda_{k+1}=O(\lambda_k),$$

and $\phi(w)$ is positive for $w\geq 0$ and approaches zero as $w\to\infty$, the process Γ is said to belong to the class $T(\lambda,\phi)$ when $\sum_{k=0}^\infty |\nu_{w,k}|<\infty$ for all finite w and $1-\nu_{w,k}\sim\lambda_k\phi(w)$ for k fixed, $w\to\infty$. The process Γ is said to be λ-convex if the sequences $(\nu_{w,k})$ and $[(\nu_{w,k}-\nu_{w,k+1})/(\lambda_{k+1}-\lambda_k)]$ are decreasing towards zero for all finite w sufficiently large. If in addition to condition (1) the sequence λ_k satisfies the condition

$$\sum_1^{n-1} k|\lambda_{k-1}-2\lambda_k+\lambda_{k+1}|+n(\lambda_n-\lambda_{n-1})=O(\lambda_n),$$

the author establishes the theorem: Let Γ be a λ-convex process belonging to the class $T(\lambda,\phi)$; the approximation of its saturation is $O[\phi(w)]$ and its class of saturation is the linear variety Σ defined by:

$$\text{in } C: f\in\Sigma\Leftrightarrow G=\sum_1^\infty \lambda_k k^{-1}\bar{A}_k(x)\in \operatorname{Lip} 1;$$

$$\text{in } L^1:\ f\in\Sigma\Leftrightarrow G\sim\sum_1^\infty \lambda_k k^{-1}\bar{A}_k(x) \text{ is of bounded variation};$$

$$\text{in } L^p(p>1): f\in\Sigma\Leftrightarrow g\sim\sum_0^\infty \lambda_k A_k(x)\in L^p.$$

D. Moskovitz (Pittsburgh, Pa.)

7183:

Sunouchi, Gen-ichiro; and Watari, Chinami. On determination of the class of saturation in the theory of approximation of functions. Proc. Japan Acad. 34 (1958), 477–481.

Let $f(x)$ be integrable of period 2π. Let $P_n(x)$ be the means of the Fourier expansion of $f(x)$ according to some linear summation method.

If there is a positive non-increasing function $\varphi(n)$ and a class K of functions in such a way that (I) $\|f(x)-P_n(x)\|=o(\varphi(n))$ implies $f(x)=\text{constant}$, (II) $\|f(x)-P_n(x)\|=O(\varphi(n))$ if and only if $f(x)\in K$, then it is said that the method is saturated with order $\varphi(n)$ and its saturation class is K. Here $\|\ \|$ indicates any of the L^p norms. This definition is due to J. Favard, although particular cases of the phenomena have been known for a long time. (For some of the recent literature see the references quoted in this paper.)

This paper contains a statement (proofs are promised for a future time) of theorems which contain the determination of the order $\varphi(n)$ and the class K for various basic methods of summation. These are: The Cesàro-Fejér method, and the methods of Abel-Poisson, Riesz (R, n^p, τ), Gauss-Weierstrass, Bernstein-Rogosinski, de la Vallée Poussin, and the method of Jackson-de la Vallée Poussin.

The underlying spaces considered are L^p, $1\leq p<\infty$, and also $C=$(continuous functions with sup norm).

E. M. Stein (Chicago, Ill.)

7184:

Karamata, J. Sur les facteurs de convergence uniforme des séries de Fourier. Rev. Fac. Sci. Univ. Istanbul Sér. A 22 (1957), 35–43. (Turkish summary)

We say that $\{\lambda_n\}\in(P, Q)$ if using λ_n as multipliers transforms every Fourier series of class P into one of class Q: the classes of interest here are C (continuous), C_F (uniformly convergent Fourier series), M (bounded and measurable), L (integrable), S (Fourier-Stieltjes). The present note summarizes results on (P, C_F) obtained by Tomić [Acad. Serbe Sci. Publ. Inst. Math. **8** (1955), 23–32; MR **17**, 963], the author [J. Math. Pures Appl. (9) **35** (1956), 87–95; MR **17**, 964] and Katayama [J. Fac. Sci. Hokkaido Univ. Ser. I **13** (1957), 121–129], and adds some further ones, principally as follows: Sufficient conditions for (P, C_F), when P is the indicated class, are: (C or M) $\lambda_n\to 0$ and $\sum_{n=1}^\infty n^{-1}\sum_{\nu=n}^\infty|\Delta\lambda_\nu|<\infty$; ($L$) $\lambda_n\to 0$ and $\sum_{\nu=n}^\infty|\Delta\lambda_\nu|=O(1/n)$ and $\sum_{\nu=0}^n \lambda_\nu=O(1)$; ($S$) $\mu_n=O(1/n)$ and $\sum_{\nu=0}^\infty \lambda_\nu$ converges.

R. P. Boas, Jr. (Evanston, Ill.)

7185:

Ciesielski, Z. On absolute convergence of Fourier series of almost all functions of Wiener space. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 501–503.

The author states six theorems dealing with the behavior of the Fourier series partial sums of almost all functions $x(t)$ in the space C of real-valued continuous functions on $[0, 1]$ with $x(0)=0$. The measure on C is in the sense of Wiener. [Acta Math. **55** (1930), 117–258]. One will suffice as an illustration. Let $\omega(t)$ be an increasing function such that $\omega(2t)/\omega(t)=O(1)$ as $t\uparrow$. Let $S_n[x]=\sum_1^n|a_k[x]|$ where $a_k[x]$ is the kth Fourier coefficient of x. Then, there are integers $N[x]$ such that, for almost all $x\in C$,

$$\sum_{n\geq N[x]}\frac{1}{n\omega\{S_n[x]\}}<\infty$$

if and only if $\sum 1/\omega(n)$ converges. The orthogonal functions are restricted to the classical trigonometric, or the Haar or Walsh functions.

R. C. Buck (Stanford, Calif.)

7186:

Herz, C. S. Spectral synthesis for the circle. Ann. of Math. (2) 68 (1958), 709–712.

On sait [L. Schwartz, C. R. Acad. Sci. Paris 227 (1948), 424–426; MR 10, 249] qu'il existe une fonction continue sur R^3, dont le spectre est porté par la sphère unité, et qui n'est pas approchable dans la topologie faible de $L^\infty(R^3)$ par des combinaisons linéaires d'exponentielles $e^{i\lambda x}$, $|\lambda|=1$. L'auteur montre qu'il n'en est pas ainsi quand on remplace R^3 par R^2 et la sphère par le cercle. La démonstration utilise à la fois l'invariance du cercle par les rotations et le fait que le cercle est de dimension 1.

J. P. Kahane (Montpellier)

7187:

Shapiro, Victor L. The conjugate Fourier-Stieltjes integral in the plane. Bull. Amer. Math. Soc. 65 (1959), 12–15.

Let x and y denote vectors (x_1, x_2) and (y_1, y_2) in two dimensions, (x, y) their scalar product, and E_2 the whole real plane. Then if

$$f(y)=(2\pi)^{-2}\int_{E_2} e^{-i(y,x)}dF(x),$$

$$k(y)=(2\pi)^{-2}\int_{E_2} e^{-i(y,x)}K(x)dx,$$

the author defines $\tilde{F}(x)$, the conjugate Fourier-Stieltjes integral of $F(x)$ with respect to the kernel $K(x)$, by

$$\int_{E_2} K(x-y)dF(y).$$

If $K(x)$ is of the form $\Omega(\theta)r^{-2}$, where r and θ are polar coordinates and $\Omega(\theta)$ is a continuous periodic function of θ of period 2π, then formally

$$\tilde{F}(x)=\lim_{R\to\infty} 4\pi^2\int_{E_2} e^{-|y|/R}e^{i(y,x)}f(y)k(y)dy.$$

The author sketches a proof of this result when $K(x)$ and $F(x)$ satisfy various conditions, and appropriate principal or Cauchy values of the integrals are used. He announces that corresponding results for n dimensions are to be published in more detail later.

A. P. Guinand (Edmonton, Alta.)

7188:

Gosselin, Richard P. A convergence theorem for double L^2 Fourier series. Canad. J. Math. 10 (1958), 392–398.

The author proves a theorem which implies the convergence of certain sub-sequences of partial sums of a double Fourier series in L^2. The author defines the notion of a double sequence of integers to have upper density one. Using this notion, a particular result detained by the author is the following. Let $f(x, y)\in L^2$. Then there exists a double sequence P of upper density one, such that almost everywhere

$$\lim_{\substack{p,q\to\infty\\(p,q)\in P}} S_{p,q}(x, y)=f(x, y).$$

The double sequence P does not depend on the point (x, y), but depends on the function $f(x, y)$ in question.

The author proves his results by obtaining refinements of similar results first obtained for functions of one variable [see Proc. Amer. Math. Soc. 7 (1956), 392–397; MR 18, 303]. These one-dimensional results are then used to prove the two-dimensional theorem quoted above.

E. M. Stein (Chicago, Ill.)

INTEGRAL TRANSFORMS

See also 7222, 7223.

7189:

Ragab, F. M. The inverse Laplace transform of an exponential function. Comm. Pure Appl. Math. 11 (1958), 115–127.

The inverse Laplace transform $f(t)$ of

$$g(p)=\int_0^\infty f(t)\exp(-pt)dt=p^{\alpha-1}\exp(-a^{1/m}p^{1/m}),$$

$\mathrm{Re}(p, \alpha, a)>0$, is established in terms of MacRobert's E function, and the asymptotic behavior of $f(t)$ for small t is given.

F. Oberhettinger (Madison, Wis.)

7190:

Narain, Roop. Certain rules of generalized Laplace transform. Gaṇita 8 (1957), 25–35.

The generalised transform discussed is

$$W[f(t); k, m]=s\int_0^\infty (st)^{m-\frac{1}{2}}e^{-st/2}W_{k,m}(st)f(t)dt.$$

The author derives formally ten formulae connecting transforms of different orders. The simplest is

$$W[f'(t); k, m]=sW[f(t); k+\tfrac{1}{2}, m-\tfrac{1}{2}]$$

(provided certain conditions hold at $t=0$).

By the use of these formulae, a number of transform images are obtained from known images. The results are expressed in terms of generalised hypergeometric functions.

J. L. Griffith (Kensington)

7191:

Sunouchi, Gen-Ichirô. Some theorems on fractional integraton. Tôhoku Math. J. (2) 9 (1957), 307–317.

This paper contains theorems which are slight variants of known results. The author is interested particularly in the type of theorems on fractional integration considered by I. I. Hirschman [Amer. J. Math. 75 (1953), 531–546; MR 15, 119].

Typical is the following generalization: Let

$$u_\alpha(\rho, \theta)\sim\sum c_n(in)^{-\alpha}\rho^n e^{in\theta}.$$

The author then considers the function

$$g^*(\alpha, \beta\, \theta)=\left\{\int_0^1 (1-\rho)^{2(\beta-\alpha)}d\rho\int_0^{2\pi}\frac{|u_{\alpha-1}(\rho, \theta+t)|^2dt}{|1-\rho e^{it}|^{2\beta}}\right\}^{\frac{1}{2}};$$

and proves various inequalities for this function similar to the known special case corresponding to $\alpha=0$.

E. M. Stein (Chicago, Ill.)

7192:

Koizumi, Sumiyuki. On fractional integration. Tôhoku Math. J. (2) 9 (1957), 298–306.

The author makes use of a result of the preceding paper [reviewed above] to obtain some theorems related to fractional integration. The main theorem of the paper, theorem 1, is a slight extension of a theorem of Hirschman [see article quoted in above review] concerning the integral

$$\int_0^{2\pi} |u_\alpha(\theta+t)-u_\alpha(\theta-t)|^2t^{-2\alpha-1}dt$$

where $u_\alpha(\theta)\sim\sum a_n(in)^{-\alpha}e^{in\theta}$.

E. M. Stein (Chicago, Ill.)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

See also 7095, 7097.

7193:

★Volterra, Vito. Theory of functionals and of integral and integro-differential equations. With a preface by G. C. Evans, a biography of Vito Volterra and a bibliography of his published works by E. Whittaker. Dover Publications, Inc., New York, 1959. iii+226 pp. (1 plate) $1.75.

An unabridged republication of the first English translation published by Blackie and Son, Ltd. in 1930. (The original Spanish edition was published by Univ. Central, Madrid, in 1927.) The preface, the biography and the bibliography have been added to this edition.

7194:

Kantorovitz, Shmuel. On the integral equation:

$$(1)\quad \varphi(x,y)-\lambda a(x,y)\int \varphi(x,y)dx-\mu b(x,y)\int \varphi(x,y)dy=c(x,y)$$

Riveon Lematematika **12** (1958), 24–26. (Hebrew. English summary)

It is shown that equation (1) corresponds to a Fredholm equation of the second kind. The solvability of (1) is shown to depend on the solvability of the Fredholm equation with the usual alternative available. Unique solutions exist for given functions $c(x, y)$ and trivial solutions for $c\equiv 0$ except for special functions, $c(x, y)$. These special functions, in turn, depend on the homogeneous Fredholm equation. *Hirsh Cohen* (Delft)

7195:

Picone, Mauro. Sullo spettro in un parametro da cui dipendono certe equazioni integrali lineari. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **23** (1957), 347–354.

The author concerns himself with certain spectral aspects of finite systems of linear integral equations:

$$(1)\quad \sum_{j=1}^{n}\int_{A_j} K_{ij}(x_i, y_j, \lambda)\phi_j(y_j)dy_j=0,\quad x_i\in A_i,\ i=1,\cdots,n;$$

or

$$(2)\quad \sum_{j=1}^{n}\int_{A_j} K_{ij}(x_i, y_j, \lambda)\phi_j(y_j)dy_j=\phi_i(x_i),\ x_i\in A_i,\ i=1,\cdots,n.$$

$K_{ij}(x_i, y_j, \lambda)$ and $\phi_i(x_i)$ are complex-valued functions defined on $A_i\times A_j\times\Gamma$ and A_i, respectively (the A_i are r_i-dimensional measurable sets and Γ is the complex plane), satisfying various natural summability conditions. Assuming some simple dependences of the K_{ij} upon λ, e.g., $K_{ij}(x_i, y_j, \lambda)=\lambda H_{ij}(x_i, y_j)$ or $H_{ij}(x_i, y_j)$ according as $i<j$ or $i\geqq j$, and $H_{ij}(x_i, y_j)=\overline{H_{ji}(y_j, x_i)}$, he obtains bounds for the λ-spectrum for problems (1) and (2).

{Incidentally, theorem I is incorrect, since in this case the spectrum of (1) need not be real; actually the spectrum does lie within $|\lambda|\geqq 1$, $\lambda=1$ excluded.}

C. R. DePrima (Pasadena, Calif.)

7196:

Calderón, A. P.; and Zygmund, A. Singular integral operators and differential equations. Amer. J. Math. **79** (1957), 901–921.

On désigne par C_β l'espace des fonctions sur E_k, $[\beta]$ fois continûment différentiables, bornées ainsi que toutes leurs dérivées d'ordre $\leqq[\beta]$, vérifiant une condition de Hölder uniforme d'ordre $\beta-[\beta]$; si $h(x, z)$ est définie sur $E_k\times E_k$, on dit que $h\in C_\beta{}^\infty$ si h est indéfiniment différentiable en z, toutes ses dérivées en z étant dans C_β comme fonctions de x.

Soit h donné dans $C_\beta{}^\infty$ avec: $h(x, \lambda z)=\lambda^{-k}h(x, z)$ pour tout $\lambda>0$, $\int_\Sigma h(x, z)d\sigma_z=0$ (Σ=sphère $|z|=1$, $d\sigma$=élément d'aire); soit a donné dans C_β; on pose:

$$H_\varepsilon f=af+\int_{|x-y|>\varepsilon} h(x, x-y)f(y)dy.$$

On a les résultats suivants: 1) l'opérateur H_ε est continu de L^p dans lui même; lorsque $\varepsilon\to 0$, $H_\varepsilon f\to Hf$ dans L^p pour tout $f\in L^p$; l'opérateur H ainsi défini est continu de L^p dans lui même; 2) H est continu de l'espace des fonctions $\in L^p$ ainsi que toutes leurs dérivées distributions d'ordre $\leqq r$, dans lui même, si $r\leqq\beta$; 3) H est continu de l'espace des fonctions L^p et Hölder continues d'ordre α, $0<\alpha<\beta$, dans lui même.

Résultats analogues pour "l'adjoint", défini à partir de

$$H_\varepsilon{}^*f=\bar{a}f+\int_{|x-y|>\varepsilon}\bar{h}(y, y-x)f(y)dy.$$

Définition du symbole $\sigma(H)$ de H: transformée de Fourier en z de $a(x)\delta(z)+h(x, z)$. Les A. montrent que toute fonction de x et z, homogène de degré 0 en z, $\in C_\beta{}^\infty$ dans $|z|\geqq 1$, est le symbole d'un opérateur H (comme ci-dessus) unique. On définit alors $H_1\circ H_2$ comme l'opérateur dont le symbole est $\sigma(H_1)\sigma(H_2)$. Evidemment si les H_i sont des opérateurs de composition (i.e. pour H, $h(x, z)$ ne dépend pas de x), $H_1\circ H_2=H_1H_2$ (composition usuelle). Dans le cas général, les A. montrent que $(H_1\circ H_2-H_1H_2)\Lambda$ et $\Lambda(H_1\circ H_2-H_1H_2)$ sont bornés de L^p dans lui même, Λ désignant une racine carrée de $-\Delta$, définie par les noyaux de composition de M. Riesz. De même $H\Lambda-\Lambda H$ est borné de L^p dans lui même. *J. L. Lions* (Nancy)

7197:

Hairullin, I. H. Some infinite sets of simultaneous linear algebraic equations solvable in closed form. Dokl. Akad. Nauk SSSR **123** (1958), 795–798. (Russian)

The infinite systems of linear equations

$$\text{(A)}\quad \begin{aligned} &x_n+\sum_{k=-\infty}^{\infty}a_{n-k}x_k=d_n\quad (n<0)\\ &x_n+\sum_{k=-\infty}^{\infty}b_{n,k}x_k=d_n\quad (0\leqq n\leqq p-1)\\ &x_n+\sum_{k=-\infty}^{\infty}c_{n-k}x_k=d_n\quad (n\geqq p)\end{aligned}$$

$$\text{(B)}\quad \begin{aligned} &x_n+\sum_{k=-\infty}^{\infty}[1+e^{2k\pi i/m}+\cdots+e^{2(m-1)k\pi i/m}]a_{n-k}x_k=d_n \quad (n<0)\\ &x_n+\sum_{k=-\infty}^{\infty}[1+e^{2k\pi i/m}+\cdots+e^{2(m-1)k\pi i/m}]c_{n-k}x_k=d_n \quad (n\geqq 0)\end{aligned}$$

are solved in closed form for $\{x_k\}\in l_2$ under specified restrictions on the coefficients and the constants d_n. Explicit details are not given beyond indicating that the principal tools used are the Laurent transformation and known methods of handling the subsequent Riemann boundary problem. *J. F. Heyda* (Cincinnati, Ohio)

7198:

Pogorzelski, W. Remarques concernant le travail "Sur l'équation intégrale singulière non linéaire et sur les propriétés d'une intégrale singulière pour les arcs non fermés". J. Math. Mech. **8** (1959), 159–160.

The author's theorems I and II in his recent paper [same J. 7 (1958), 515–532; MR 20 #4753] are shown to be true with a weaker hypothesis. The arcs involved need have only continuous tangents at each point. The exponent β in the author's inequality (7) may be replaced by α. These two conditions replace those given in the original version. *M. S. Robertson* (New Brunswick, N.J.)

FUNCTIONAL ANALYSIS

See also 7039, 7084, 7091, 7126, 7162, 7180, 7539.

7199:

Klee, V. L., Jr.; and Long, R. G. On a method of mapping due to Kadeč and Bernstein. Arch. Math. 8 (1957), 280–285.

Using a method of mapping in terms of deviation-sequences of Bernstein, recently introduced by Kadeč [Dokl. Akad. Nauk. SSSR 92 (1953), 465–468; MR 15, 535], the authors are able to sharpen Kadeč's result and consequently are able to prove that all $\aleph_0$-dimensional (cardinality of Hamel basis) normed linear spaces are homeomorphic. *L. Brown* (Detroit, Mich.)

7200:

Ricabarra, R.; et Zarantonello, E. H. Topologies minimales dans les espaces vectoriels topologiques. Rev. Mat. Cuyana 1 (1955), 181–185.

A hausdorff linear topology τ for a linear space E is called minimal provided E admits no other hausdorff linear topology which is less fine than λ. The authors prove that a locally convex hausdorff linear topology τ is minimal if and only if the space (E, τ) is isomorphic with a product of copies of the scalar field. They ask whether the assumption of local convexity can be avoided and, in particular, whether, in the space of measurable functions on [0, 1], the topology of convergence in probability is minimal. *V. L. Klee, Jr.* (Copenhagen)

7201:

Green, H. F. Convergence in sequence spaces. Proc. Edinburgh Math. Soc. 11 (1958/59), 83–85.

For a linear space α of complex sequences $x=(x_1, x_2, \cdots)$, let α^* denote the set of all complex sequences $u=(u_1, u_2, \cdots)$ such that the series $ux=\sum x_i u_i$ is absolutely convergent for all $x \in \alpha$. The space α is called perfect provided $\alpha^{**}=\alpha$. A sequence $x^{(1)}, x^{(2)}, \cdots$ in α is projectively convergent [to the limit x] provided there exists $\lim_{n\to\infty} ux^{(n)}$ $[=x]$ for all $u \in \alpha^*$. (In a perfect space, every projectively convergent sequence has a limit.) A set $X \subset \alpha$ is projectively bounded provided $\sup_{x\in X} |ux|$ is finite for each $u \in \alpha^*$, and a sequence $x^{(1)}, x^{(2)}, \cdots$ is strongly projectively convergent [to x] provided for each projectively bounded $U \subset \alpha^*$ and each $\varepsilon>0$, $|ux^{(p)}-ux^{(q)}|<\varepsilon$ for all $u \in U$ and $p, q>M(\varepsilon, U)$ $[|ux^{(n)}-ux|<\varepsilon$ for all $u \in U$ and $n>N(\varepsilon, U)]$. The author's theorem is as follows: If a perfect sequence space α is normed in such a way that a sequence is projectively convergent if and only if it is norm-convergent, then projective convergence, norm convergence, and strong projective convergence are all the same, as are also the relevant limits. *V. L. Klee, Jr.* (Copenhagen)

7202:

Bauer, Heinz. Minimalstellen von Funktionen und Extremalpunkte. Arch. Math. 9 (1958), 389–393.

A well-known formulation of the Krein-Milman theorem asserts that if X is a compact convex subset of a locally convex hausdorff linear space E, then every continuous linear functional on E attains its minimum on X at some extreme point of X [Bourbaki, Espaces vectoriels topologiques, Chaps. I–II, Actualités Sci. Ind. no. 1189, Hermann, Paris, 1953; MR 14, 880]. The author extends this result, proving that if X is a nonempty compact set (in a locally convex space) and f a lower semicontinuous concave function on X to $]-\infty, \infty]$, then f attains its minimum at some extreme point of X. (Here an extreme point of X is one not in any open segment contained in X, and f is concave provided its restriction to every segment in X is concave in the usual sense.) For special X and E, this extension was stated by Rosenbloom [Bull. Soc. Math. France 79 (1951), 1–58; MR 15, 223], but the author observes that Rosenbloom's proof was incorrect. The author's extension is an easy consequence of his principal result, which is purely topological although its proof resembles the proof of the Krein-Milman theorem given by J. L. Kelley et al [Kelley, J. Osaka Inst. Sci. Tech. Part I 3 (1951), 1–2; MR 13, 249]: Suppose X is a nonempty compact space and $\mathcal{S}$ is a system of subsets of X, each including at least two points. Let $\mathcal{F}$ denote the set of all lower semicontinuous functions f on X to $]-\infty, \infty]$ such that whenever f attains its minimum on a set $S \in \mathcal{S}$ at some point of S, then it is constant on S. Then if the points of X are distinguished by the members of $\mathcal{F}$, each $f \in \mathcal{F}$ must attain its minimum on X at at least one point not included in any member of $\mathcal{S}$. {The reference on p. 391 should be to E. Artin rather than R. Arens.}

V. L. Klee, Jr. (Copenhagen)

7203:

Wada, Junzo. Strict convexity and smoothness of normed spaces. Osaka Math. J. 10 (1958), 221–230.

The following were among questions raised by M. M. Day in his study of strict convexity and smoothness of normed linear spaces [Trans. Amer. Math. Soc. 78 (1955), 516–528; MR 16, 716]. Is every L-space strictly convexifiable? Is there a nonreflexive nonseparable Banach space which is both strictly convex and smooth? More recently [Proc. Amer. Math. Soc. 8 (1957), 415–417; MR 19, 868], Day himself has answered the first question affirmatively. In the present paper the author obtains this same affirmative answer for certain classes of L-spaces, but not for all. (His work was independent of Day's.) He also observes that an affirmative answer to the second question is provided by the suitably normed product of an arbitrary non-reflexive separable Banach space by one which is smooth, strictly convex, and nonseparable. He studies the smoothness and strict convexity of spaces of continuous functions, and concludes, in particular, that if R is metric, then CR is smoothable if and only if R is compact, and CR is strictly convexifiable if and only if R is separable.

V. L. Klee, Jr. (Copenhagen)

7204:

Andô, Tsuyoshi. On the structure of the associated modular. Proc. Japan Acad. 34 (1958), 587–588.

For a vector lattice R we have two kinds of conjugate spaces: the one, $\bar{R}$, consists of all bounded linear functionals on R and the other, $\bar{\bar{R}}$, of all universally continuous linear functionals on R [cf. H. Nakano, Jap. J. Math.

17 (1941), 425–511; MR 3, 210]. $\bar{R}$ is a normal manifold of $\bar{\bar{R}}$, and putting $\bar{\bar{R}}=\bar{\bar{R}}\oplus\bar{R}^{\perp}$, $\bar{R}^{\perp}$ is called the singular part of $\bar{\bar{R}}$. In the case where R is a modulared vector lattice [cf. H. Nakano, *Modulared semi-ordered linear spaces*, Maruzen, Tokyo, 1950; MR 12, 420], the author proved that the singular part $\bar{R}^{\perp}$ is an L_1-space in the generalized sense. This result is considered a big contribution in the sense that the structure of $\bar{R}$ determines that of $\bar{\bar{R}}$ for a modulared vector lattice R. What happens for more general spaces, for instance, normed vector lattices, remains an interesting problem. *H. Nakano* (Sapporo)

7205:

Koshi, Shôzô. On some type of the modulared linear space. J. Fac. Sci. Hokkaido Univ. Ser. I 14 (1958), 16–28.

The author defines some uniform properties concerning values of the modular in a modulared semi-ordered linear space [cf. H. Nakano, *Modulared semi-ordered linear spaces*, Maruzen, Tokyo, 1950; MR 12, 420] at neighbourhoods of zero and infinity, and shows some immediate consequences, for instance duality, among them. These properties of modulars (without uniformity) were studied by H. Nakano in his book cited above.

I. G. Amemiya (Kingston, Ont.)

7206:

Koshi, Shōzō. On semi-continuity of functionals. I. Proc. Japan. Acad. 34 (1958), 513–517.

The author extends a known fact concerning singular linear functionals on totally continuous semi-ordered linear spaces to slightly more general functionals. This is a vector lattice version of the following simple fact: Let K be a compact set in which every set of first category is nowhere dense; then for every measure on K, there exists a dense open set G such that every nowhere dense set in G is of measure 0. *I. G. Amemiya* (Kingston, Ont.)

7207:

Sasaki, Masahumi. On certain properties of modular convergence. J. Fac. Sci. Hokkaido Univ. Ser. I 14 (1958), 37–49.

In this paper certain theorems about certain types of convergence of sequences of functions in Orlicz spaces, which were recently given by W. A. J. Luxemburg and A. C. Zaanen [Indag. Math. 18 (1956), 217–228; MR 17, 1113], are extended to modulared semi-ordered linear spaces. For details we refer to the paper.

W. A. J. Luxemburg (Pasadena, Calif.)

7208:

Fréchet, M. Sur deux problèmes d'analyse non résolus. Colloq. Math. 6 (1958), 33–40.

The author proposes two problems about spaces that occur in analysis.

Let "space" mean a set with a closure operator on subsets e that is required only to satisfy the condition $e\subset\bar{e}$, and let "homeomorphism" be defined in the usual way in terms of this closure operator. For subsets A and B of two spaces, let $dA\leq dB$ mean that A is homeomorphic to a subset of B. If $dA=dB$, then either A and B are homeomorphic, or there exist decompositions $A=A_1+A_2$, $B=B_1+B_2$ such that A_k and B_k $(k=1, 2)$ are homeomorphic [Banach, Fund. Math. 6 (1924), 236–239]. The fact that many important pairs of spaces A, B in analysis (e.g., L_2 and C) satisfy the condition $dA=dB$ suggests problem 1: If A and B are homeomorphic, specify a homeomorphism. Otherwise, specify decompositions and corresponding homeomorphisms as above.

Problem 2 is an independent question: Can the set of continuous curves in R_3 be made into a Banach space in a natural way? (A similar problem can be stated about surfaces.) Remarks are made about the definition of addition, which is the difficult step, and it is shown that the problem can be reduced to finding a suitable definition for the addition of oriented polygonal arcs.

C. W. Kohls (Urbana, Ill.)

7209:

Hasumi, Morisuke. The extension property of complex Banach spaces. Tôhoku Math. J. (2) 10 (1958), 135–142.

It is known that a real Banach space B has the extension property (any bounded linear map T of a subspace E of a Banach space F into B has an extension T' with domain F and with $\|T\|=\|T'\|$) if and only if B is equivalent to the space of all continuous real-valued functions on a Stonian space [see Nachbin, Trans. Amer. Math. Soc. 68 (1950), 28–46; MR 11, 369; Goodner, ibid. 69 (1950), 89–108; MR 12, 266; and Kelley, ibid. 72 (1952), 323–326; MR 13, 659]. The author proves the corresponding theorem for complex Banach spaces. The proof requires a good bit of technique, and is based on the preliminary theorem: If χ is an upper semicontinuous mapping of a Stonian space X into the class of closed subsets of a compact Hausdorff space Y, then there is a continuous selection function f (i.e., a continuous function from X to Y such that $f(x)\in\chi(x)$ for all x in X).

J. L. Kelley (Berkeley, Calif.)

7210:

Weston, J. D. A Banach space which is not equivalent to an adjoint space. Proc. Edinburgh Math. Soc. 11 (1958/59), 105.

It is easily seen that the unit cell of the Banach space (c_0) has no extreme points, and thus from the Kreĭn-Milman extreme point theorem it follows that (c_0) is not equivalent to the adjoint of any Banach space. The author supplies a more elementary argument which has this same conclusion. *V. L. Klee, Jr.* (Copenhagen)

7211:

Foguel, S. R. Biorthogonal systems in Banach spaces. Pacific J. Math. 7 (1957), 1065–1072.

Given a separable Banach space B, the system (x_n, f_n) is called a biorthogonal system if $x_n\in B$, $f_n\in B^*$ and $f_n(x_m)=\delta_{nm}$. The author presents necessary and sufficient conditions for the sequence $\{x_n\}$ to constitute a Schauder basis in B. For the case where B is a Hilbert space, more extensive results are derived.

L. Brown (Detroit, Mich.)

7212:

Perfect, Hazel. Pythagorean orthogonality in a normed linear space. Proc. Edinburgh Math. Soc. (2) 9 (1958), 168–169.

The author gives a short proof of the following theorem due to R. C. James [Duke Math. J. 12 (1945), 291–302; MR 6, 273]. Given a Banach space T with the property: Whenever vectors x and y in T satisfy

$$\|x\|^2+\|y\|^2=\|x-y\|^2,$$

then, for all scalars λ, μ,

$$\|\lambda x\|^2+\|\mu y\|^2=\|\lambda x-\mu y\|^2.$$

Then T is a Hilbert space. This is proved by showing that all vectors x and y in T satisfy the parallelogram identity:

$$\|x-y\|^2+\|x+y\|^2=2\|x\|^2+2\|y\|^2.$$

E. R. Lorch (New York, N.Y.)

7213:

Kaniel, Shmuel. On convex sets in Hilbert space. Riveon Lematematika **12** (1958), 19–23. (Hebrew. English summary)

By an elementary argument involving projections of the sets under consideration, the author proves the well-known fact that if a family of bounded closed convex subsets of Hilbert space has the finite intersection property, then there is a point common to all the sets in the family. Though he claims to employ Helly's theorem on the intersection of convex sets, his proof appears instead to involve the compactness of bounded closed subsets of Euclidean space. He proves also the known result that if a subset of Hilbert space is of diameter ≤ 1, then it lies in a sphere of diameter $\leq \sqrt{2}$. This has been noted before by Routledge [Quart. J. Math. Oxford Ser. (2) 3 (1952), 12–18; MR **13**, 661], and extended in the review of Routledge's paper [loc. cit.].

V. L. Klee, Jr. (Copenhagen)

7214:

Anderson, Frank W.; and Blair, Robert L. Characterizations of the algebra of all real-valued continuous functions on a completely regular space. Illinois J. Math. **3** (1959), 121–133.

Let $C(X)$ denote the set of all continuous real-valued functions on a completely regular Hausdorff space X. The main results of the paper consist of statements of the form: If a set A satisfies certain conditions that can be stated in terms of algebraic and order concepts, then A is some type of subset of $C(X)$ for a certain kind of space X; together, in most cases, with a converse. Except in one theorem, the space X is determined by A up to homeomorphism.

The set A is viewed, in turn, as a ring, an algebra over the real field R, a lattice-ordered algebra, and a vector lattice. The space X is usually either a compact space or a Q-space [Hewitt, Trans. Amer. Math. Soc. **64** (1948), 45–99; MR **10**, 126]. The subring of $C(X)$ to which A is isomorphic always distinguishes points of X. Included among the results are several algebraic characterizations of all of $C(X)$, for X a compact space or a Q-space (the latter actually being no restriction in this case).

The special algebraic and order conditions imposed on A involve the set $\mathfrak{R}_A$ of kernels of the homomorphisms of A onto R, and the sets of images of elements of A under some family of these homomorphisms. The hypothesis used to ensure that A is isomorphic to all of $C(X)$ is, essentially, that A is isomorphic to every extension B satisfying the same other conditions as A, and such that the mapping $M \rightarrow M \cap A$ from $\mathfrak{R}_B$ onto $\mathfrak{R}_A$, with appropriate topologies, is a homeomorphism.

C. W. Kohls (Urbana, Ill.)

7215:

Arsove, Maynard G. Proper bases and linear homeomorphisms in spaces of analytic functions. Math. Ann. **135** (1958), 235–243.

V. Ganapathy Iyer introduced the notion of a "proper basis" in the space of entire functions. In a previous paper [Proc. Amer. Math. Soc. **8** (1957), 264–271; MR **19**, 259] the author solved the problem of characterization of automorphisms by means of bases. In the present paper the author extends this theory to spaces of functions analytic on a neighbourhood $N_R(0)$ of radius R about the origin. Proper bases are defined so as to preserve the essential properties of coefficient sequences for expansions in terms of the fundamental basis $\{\delta_n\}:\delta_n(z)=z^n$ ($n=0, 1, 2, \cdots$) (power series). Proper bases are then characterized by the limiting behaviour of the maximum function. Linear homeomorphic mappings onto closed subspaces are characterized as the linear mappings carrying $\{\delta_n\}$ into proper bases.

B. A. Amira (Jerusalem)

7216:

Arsove, Maynard G. Similar bases and isomorphisms in Fréchet spaces. Math. Ann. **135** (1958), 283–293.

The author examines the relation between bases and linear homeomorphisms in Fréchet spaces (metrizable, complete, locally convex topological linear spaces—real or complex). He thus generalizes results which he has obtained previously for spaces of analytic functions [#7215 above] and relates them to earlier work of the reviewer in Hilbert space [Bull. Amer. Math. Soc. **45** (1938), 564–569]. If U and V are vector spaces and if $\{x_n\}$ is a basis in U and T is a linear homeomorphism between U and V, then $\{y_n=Tx_n\}$ is clearly a basis in V. The very substantial converse problem is the subject of the present paper. A sequence $\{x_n\}$ in U is said to be similar to a sequence $\{y_n\}$ in V provided that $\sum_{n=1}^\infty a_n x_n$ converges $\Leftrightarrow$ $\sum_{n=1}^\infty a_n y_n$ converges. The fundamental theorem then is that if $\{x_n\}$ and $\{y_n\}$ are bases in the Fréchet spaces U and V, respectively, and if they are similar, then there exists a linear homeomorphism (isomorphism) T of U on V such that $Tx_n=y_n$.

The Fréchet space structure is introduced by means of an increasing sequence of semi-norms $\{u_n\}$ for U and $\{v_n\}$ for V. In terms of these, three conditions relating pairs of sequences $\{x_n\}$ and $\{y_n\}$ are given. These conditions are proved to be equivalent in the case of bases and are necessary and sufficient for the existence of an isomorphic map of U on V. One such condition is reproduced here: For each p there exist q and j (all positive integers) and $M>0$ such that for all sequences $a_j, a_{j+1}, \cdots, a_k$,

$$v_p\left(\sum_{n=j}^{k} a_n y_n\right) \leq M \sup_{j\leq t\leq k} u_q\left(\sum_{n=j}^{t} a_n x_n\right).$$

There is also a discussion, roughly parallel to the preceding, which treats the case of absolutely convergent bases. Such bases are defined by: $\sum_{n=1}^\infty u_q(a_n x_n)<\infty$, $q=1, 2, \cdots$, whenever $\sum_{n=1}^\infty a_n x_n$ converges. Two sequences $\{x_n\}$ and $\{y_n\}$ are said to be absolutely similar if $\sum_{n=1}^\infty v_p(a_n y_n)<\infty$, $p=1, 2, \cdots$, $\Leftrightarrow \sum_{n=1}^\infty u_q(a_n x_n)<\infty$, $q=1, 2, \cdots$. The principal result is that if two absolutely convergent bases $\{x_n\}$ and $\{y_n\}$ are absolutely similar, then there is an isomorphism T of U on V such that $Tx_n=y_n$.

E. R. Lorch (New York, N.Y.)

7217:

Bishop, Errett. Spectral theory for operators on a Banach space. Trans. Amer. Math. Soc. **86** (1957), 414–445.

The spectral theory of a closed linear operator T in a reflexive B-space B is studied by means of vector-valued measures m (defined on the Borel sets of the plane and countably additive) called T-measures and related to T by the requirement that $Tm(S)=\int_S z\,dm(z)$ for all bounded Borel sets S. In case T is normal the only T-measures are of the form $m(S)=E(S)x$, where E is the resolution of the identity for T. Let $\bar{G}_c$ be the closure in the strong operator topology of the set G_c of all scalar operators T with $\sup |f(T)x|\leq c$, where the supremum is taken over all bounded measurable functions f with $|f|_\infty\leq 1$ and all x in B with $|x|\leq 1$. Then for every T in $\bar{G}_c$ there is a mapping $S\rightarrow E(S)$ of the Borel sets in the plane into the operators on B whose properties include the following: for x in

$D(T)$, $Tx=\int z\,dE(z)x$; if T is in the closure of the subset of G_c consisting of those operators whose spectrum is in a given closed set C, then $E(S)=0$ if S is disjoint from C; if B is Hilbert space and $c=1$, then $E(S)$ is a positive Hermitian operator. Among the several corollaries are the spectral theorems for symmetric transformations and self-adjoint transformations.

N. Dunford (Brooklyn, N.Y.)

7218:

Foiaş, Ciprian. La mesure harmonique-spectrale et la théorie spectrale des opérateurs généraux d'un espace de Hilbert. Bull. Soc. Math. France **85** (1957), 263–282.

Let S be a spectral set in the sense of J. von Neumann for the bounded linear operator T in Hilbert space H. It is supposed that the frontier of S is a simple closed curve. The symbol $\mathfrak{A}$ [resp. $\mathfrak{O}$] denotes the family of functions $u(\lambda)$ harmonic [resp. $f(\lambda)$ holomorphic] in a domain D_u [resp. D_f] with $D_u\supset S$ $[D_f\supset S]$. The symbol $\bar{\mathfrak{A}}$ [resp. $\bar{\mathfrak{O}}$] denotes the set of functions continuous on S and harmonic [resp. holomorphic] in the interior of S. A functional calculus for functions u in $\mathfrak{A}$ is established by defining $u(T)=\operatorname{Re} f(T)$, where f is in $\mathfrak{O}$ and $u(\lambda)=\operatorname{Re} f(\lambda)$. Here $\operatorname{Re} f(T)$ is defined as $\frac{1}{2}(f(T)+f(T)^*)$. According to a result of M. Brelot the class $\bar{\mathfrak{A}}$ $[\bar{\mathfrak{O}}]$ is the closure of $\mathfrak{A}$ $[\mathfrak{O}]$ under the norm $|u|=\sup_{\lambda\in S}|u(\lambda)|$, and this fact together with a theorem of E. Heinz enables one to define the operators $u(T)$ and $f(T)$ for $u\in\bar{\mathfrak{A}}$, $f\in\bar{\mathfrak{O}}$ by placing $u(T)=\lim_n u_n(T)$, $f(T)=\lim f_n(T)$, where u_n, f_n are sequences in $\mathfrak{A}$, $\mathfrak{O}$, respectively, with $|u_n-u|\to 0$, $|f_n-f|\to 0$. Besides the usual algebraic identities for an operational calculus one has the following inequalities for $u\in\bar{\mathfrak{A}}$, $f\in\bar{\mathfrak{O}}$. $|f(T)|\leqq|f|$, $|u(T)|\leqq|u|$, $\inf_{\lambda\in S} u(\lambda)\leqq\inf_{|x|=1}\langle u(T)x, x\rangle\leqq\sup_{|x|=1}\langle u(T)x, x\rangle\leqq\sup_{\lambda\in S} u(\lambda)$. Furthermore, if $|x_n|=1$ and $(\lambda I-T)x_n\to 0$, then $\langle u(T)x_n, x_n\rangle\to u(\lambda)$. Spectral mapping theorems are proved, e.g., $\sigma[f(T)]=f(\sigma[T])$, $\sigma_p[f(T)]=f[\sigma_p(T)]$, and if $\lambda\in\sigma_p(T)$ and $E_\lambda[T]$ is the orthogonal projection of H onto the space of eigenvalues corresponding to λ, then $E_\lambda[T]=E_{f(\lambda)}[f(T)]$. The operational calculus may be extended further, and this is done by establishing the existence of the "harmonic-spectral measure" of T. This is an operator-valued measure $\omega(\beta)=\omega(T;\beta,S)$ defined on the Borel subsets β of the frontier F of S. For every bounded Borel function ψ on F, let the operator $u_\psi(T)$ be defined by the integral $\int_F\psi(\zeta)\omega(d\zeta)$. Then the mapping $\psi\to u_\psi(T)$ is a positive linear map with the property that if ψ is continuous, then $u_\psi(T)=u(T)$, where u is the uniquely defined element of $\bar{\mathfrak{A}}$ which coincides with ψ on F. A number of interesting properties of the measure ω are established. For example, the operator T is normal and $\sigma(T)\subset F$ if and only if the operators $\omega(\beta)$ are projections. In this case ω is the usual spectral measure for T. The measure ω allows the original operational calculus defined on $\bar{\mathfrak{O}}$ to be extended to certain bounded functions analytic on the interior of S and having a finite number of discontinuities on the boundary F. The theory is applied by proving a theorem of Yosida and Hille. *N. Dunford* (Brooklyn, N.Y.)

7219:

Leżański, T. Formula for the spectral decomposition. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 689–690.

Let A be a bounded self-adjoint operator on a Hilbert space H, E_λ its spectral resolution, and z^0 in H. The author approximates the function $E_\lambda(z^0)$ by smoother functions of λ. Theorem 2 then says that if A_n converges uniformly to A (where the A_n are also bounded and self-adjoint) and if ψ satisfies a Lipschitz condition, then $\psi(A_n)$ converges strongly to $\psi(A)$. {However, it is clear by the Weierstrass approximation theorem that the convergence is uniform for an arbitrary continuous ψ.}

E. Nelson (Princeton, N.J.)

7220:

Fage, M. K. A self-adjoint operator and absolutely monotone functions. Černivec. Derž. Univ. Nauk Zap. Ser. Fiz.-Mat. Nauk **4** (1952), no. 2, 159–162. (Russian)

The author gives a proof of the spectral theorem for a bounded self-adjoint operator on a Hilbert space by means of the Hausdorff-Bernstein-Widder representation theorem for completely monotone functions. The proof proceeds along the same lines as the extant proofs which use the Herglotz-Bochner representation theorem for positive definite functions or the representation theorem for Hausdorff moment sequences.

A. Devinatz (St. Louis, Mo.)

7221:

Brodskiĭ, M. S.; and Livšic, M. S. Spectral analysis of non-self-adjoint operators and intermediate systems. Uspehi Mat. Nauk (N.S.) **13** (1958), no. 1(79), 3–85. (Russian)

This paper contains a detailed reworking of some of the results of the second author's previous paper [Mat. Sb. N.S. **34(76)** (1954), 145–199=Amer. Math. Soc. Transl. (2) **5** (1957), 67–114; MR **16**, 48; **18**, 748], together with some new results and an application of this work to the quantum theory of scattering.

Here the authors restrict themselves to a bounded operator A on a Hilbert space H with $\dim(i^{-1}(A-A^*)H)<\infty$, although they remark that the generalization to the case where $i^{-1}(A-A^*)$ is completely continuous presents no difficulties. To exploit the definition of "simple relative to a subspace containing $i^{-1}(A-A^*)H$", the definition of characteristic matrix function has been changed: If $g_1, g_2, \cdots, g_r$ are vectors with span E such that $i^{-1}(A-A^*)f=\sum_{\alpha,\beta=1}^{r}(f, g_\alpha)j_{\alpha\beta}g_\beta$ for any $f\in H$, where $J=[j_{\alpha\beta}]$ is a Hermitian matrix satisfying the equation $J^2=I$, then a characteristic matrix function for A is

$$w(\lambda)=I-[((A-\lambda)^{-1}g_\alpha, g_\beta)]J.$$

If A is simple relative to E (which must contain $i^{-1}(A-A^*)H$), i.e., there is no non-trivial subspace orthogonal to E which is invariant under A, then the characteristic matrix function $w(\lambda)$ determines A up to unitary equivalence. There is also considerably more detail on the representation of characteristic matrix functions by ordered infinite products and multiplicative integrals and an appendix devoted to the latter.

A subject not dealt with in the paper referred to above is that of dissipative operators. The authors naturally restrict themselves to a class of dissipative operators which fall within the scope of the previous analysis — those operators for which $i^{-1}(A-A^*)\geqq 0$ and $\dim[i^{-1}(A-A^*)H]<\infty$. For these operators, one result is that if A is a simple dissipative operator, then the system of finite-dimensional invariant subspaces of A is dense in H if and only if $\sum\operatorname{Im}\mu_k=\operatorname{sp}[i^{-1}(A-A^*)]$, where $\{\mu_k\}_1^\infty$ are the eigenvalues of A.

R. R. Kemp (Kingston, Ont.)

7222:

Sahnovič, L. A. Reduction of a non-selfadjoint operator with continuous spectrum to diagonal form. Uspehi Mat. Nauk **13** (1958), no. 4(82), 193–196. (Russian)

The author considers linear transformations T of

$L_2=L_2[0,l]$ ($l<\infty$) into itself given by $(Tf)(x)=\alpha(x)f(x)+i\varepsilon\int_0^x f(t)dt$, where $\varepsilon=\pm 1$ and $\alpha(x)$ is nondecreasing. Livšic's theory of triangular models of non-selfadjoint operators in Hilbert space [see #7221 and reference therein, and the present author's paper, #7223 below] lead naturally tothis kind of operator as an important typical special case. The author shows that $(Af)(x)=xf(x)+i\int_0^x f(t)dt$ is similar to $(Qf)(x)=xf(x)$ in the sense that there exists a bounded linear transformation B of L_2 into itself with the bounded linear inverse B^{-1} such that $BB^{-1}=B^{-1}B=$identity transformation and $BAB^{-1}=Q$, where B is given by the fractional integral of pure imaginary order i:

$$(Bf)(x)=(1/\Gamma(i+1))(d/dx)\int_0^x (x-y)^i f(y)dy.$$

The paper by Herman Kober [Trans. Amer. Math. Soc. **50** (1941), 160–174; MR **3**, 39] is not used in the proof. It should be noted that the present result is a special case of a theorem in the paper below.

G. K. Kalisch (Minneapolis, Minn.)

7223:

Sahnovič, L. A. Reduction to diagonal form of non-self-adjoint operators with continuous spectrum. Mat. Sb. N.S. **44(86)** (1958), 509–548. (Russian)

The author considers two relations between bounded linear operators A in a Hilbert space H and A_1 in a Hilbert space H_1: (1) A and A_1 are called linearly similar if there exists a closed and densely defined linear operator B mapping H in a 1-1 fashion onto a dense subset of H_1 such that, for all h in the domain of definition D of B, Ah is also in D and $A_1Bh=BAh$; (2) A and A_1 are called linearly equivalent if they are linearly similar and if B and B^{-1} are everywhere defined and bounded (these concepts were also considered by Kurt Friedrichs [see Math. Ann. **115** (1938), 249–272 and Comm. Appl. Math. **1** (1948), 361–406; MR **10**, 547], who also proved theorems related to those of the present author). The operators considered here are special cases of operators of class $(i\Omega)$ introduced by M. S. Livšic [see #7221 above and reference therein], viz., $Af(x)=xf(x)-i\int_a^x f(t)p(t)jp(x)dt=Qf(x)-iT_{p,j}f(x)$, defined for $f\in L_2[a,b]$, where $j=1$ or -1 and the nonnegative real-valued function $p(x)$ is bounded and measurable. The first principal result is that, under these hypotheses, A is linearly similar to Q. The operator B is given by the formula

$$B\varphi(x)=(2\pi)^{-1}\frac{d}{dx}\int_a^x \frac{\varphi(\sigma)}{p(x)}(\exp(\pi p^2(\sigma))-\exp(-\pi p^2(\sigma)))^i \times\exp\left(-i\int_a^x\left(\frac{p^2(s)j}{(s-\sigma)}\right)ds\right)\,d\sigma,$$

which is related to the characteristic matrix function of A [see #7221]. The second part of the paper contains the result that if $p^2(x)$ is of bounded variation and Lip α ($0<\alpha\leq 1$), then B is actually bounded and A and Q are linearly equivalent. The special case $p\equiv 1$ was considered by the author in #7222 above, in which case B reduces to (a multiple of) fractional differentiation of order i [in this connection see Herman Kober, reference in review above]. In the third part several examples are carried out in detail, such as resolvents of certain differential operators and others. The author points out that his methods yield results even when the operator in question is no longer of class $(i\Omega)$; for example, if $A=Q-T_{p,j}$, where $p(x)$ is Lipschitz and $0<\delta\leq p(x)\leq q<1$, then A is linearly similar to Q, where B is given by

$$B\varphi(x)=(1/\pi p(x))\frac{d}{dx}\int_a^x \varphi(\sigma)\sin(\pi p^2(\sigma))\exp\left(\int_a^x\left(\frac{p^2(s)}{(s-\sigma)}\right)ds\right)d\sigma.$$

The relation between B and B^{-1} generalizes formulas connected with Abel's integral equation (for which the author's $p\equiv c$); see R. Courant and D. Hilbert, Methoden der mathematischen Physik, vol. 2, Springer, Berlin, 1937, p. 414.

G. K. Kalisch (Minneapolis, Minn.)

7224:

Nelson, Edward. Representation of a Markovian semi-group and its infinitesimal generator. J. Math. Mech. **7** (1958), 977–987.

Let $\mathfrak{C}$ be the space of all continuous functions, vanishing at infinity, on a locally compact Hausdorff space X, with the supremum norm. Let $\{P^t,\ 0\leq t<\infty\}$ be a family of linear transformations on $\mathfrak{C}$, satisfying $P^tf\geq 0$ if $f\geq 0$, $P^{t+s}=P^tP^s$, $\|P^tf\|\leq\|f\|$. Let $\mathfrak{B}$ be the closed linear invariant under every P^t subspace of those elements f with $\lim_{t\to 0}\|P^tf-f\|=0$. The P^t semigroup is strongly continuous on $\mathfrak{B}$. Let $\mathfrak{A}$ be the set of all f in $\mathfrak{C}$ for which f and f^2 are in $\mathfrak{B}$. It is shown that $\mathfrak{A}$ is a closed but not necessarily invariant algebra.

There is a regular Borel measure $p^t(x,\cdot)$ such that $P^tf\cdot(x)=\int_X f(y)p^t(x,dy)$, where $p^t(x,X)\leq 1$. If there is equality for all x ('Markovian' case), the measure $p^t(x,\cdot)$ is the transition measure of a Markov stochastic process. Moreover equality can always be effected by a suitable modification of the algebra $\mathfrak{C}$.

The infinitesimal generator A is represented in terms of a family of linear functionals defined at each point of X. This representation, which holds only in a certain algebra of functions in the domain of the infinitesimal operator of the semigroup, is the analogue of one due to Hunt [Trans. Amer. Math. Soc. **81** (1956), 264–293; MR **18**, 54] in the spatially homogeneous case.

J. L. Doob (Urbana, Ill.)

7225:

Vulih, B. Z. Partial order in rings of bounded self-adjoint operators. Vestnik Leningrad. Univ. **12** (1957), no. 13, 13–21. (Russian. English summary)

We consider the set of all bounded self-adjoint operators in a Hilbert space as partially ordered with the usual definition of positive operators. It is known that this set is even not a lattice.

In § 1 we prove quite shortly that every strongly closed ring **A** of bounded self-adjoint operators is a K-space. Furthermore, the principal unit E of the ring **A** can be taken in **A** as a unit of the K-space. In this case **A** will be a K-space of bounded elements and its base will consist of all projectors belonging to **A**.

In § 2 we obtain the important Neumann's theorem about functions of self-adjoint operators in a Hilbert space as a simple consequence of a general theorem about functions of elements in a K-space.

In § 3 we give an elementary proof of the existence of the principal unit in every strongly closed ring of bounded self-adjoint operators. *Author's summary*

7226:

Takesaki, Masamichi. On the direct product of W^*-factors. Tôhoku Math. J. (2) **10** (1958), 116–119.

The author generalizes the results of Nakamura [same J. **6** (1954), 205–209; MR **16**, 1126], which give conditions that two subfactors generate a factor. He removes the finiteness restriction, but adds a condition involving existence of σ-weakly continuous linear functionals.

E. L. Griffin, Jr. (Ann Arbor, Mich.)

7227:

Takesaki, Masamichi. On the conjugate space of operator algebra. Tôhoku Math. J. (2) **10** (1958), 194–203.

Several known results are reproven, and some new ones proven, concerning linear functionals and homomorphisms of W^*-algebras. A main tool is the splitting of the functionals or homomorphisms into singular and σ-weakly continuous parts. Let M be a W^*-algebra. φ in M^* is called singular if, writing it canonically as $\varphi_1-\varphi_2+i(\varphi_3-\varphi_4)$, $\varphi_j\geqq 0$, no φ_j is $\geqq$ any nonzero member of M_* (the σ-weakly continuous elements of M^*). It is shown that any closed subspace of M^* which is invariant under left action by elements of M is the direct sum of its part in M_* and its singular part. A bounded linear map θ from M to a W^*-algebra N is called singular if $\theta^*(N_*)$ contains only singular elements of M^*. It is shown that if θ is a $*$-homomorphism, then there exists a central projection z in the weak closure of $\theta(M)$ such that $x\rightarrow\theta(x)z$ is σ-weakly continuous and $x\rightarrow\theta(x)(1-z)$ is singular. If M, N are σ-finite and θ is an isomorphism, then the singular part is not faithful, and the other part is. Also properties of M are derived from properties of maximal abelian subalgebras A, for example $K\subset M_*$ is relatively $\sigma(M, M_*)$ compact $\Leftrightarrow$ $K|A$ is relatively $\sigma(A, A_*)$ compact for all maximal abelian A. *J. Feldman* (Berkeley, Calif.)

7228:

Naĭmark, M. A. On the expansion of the tensor product of representations of the principal series of the proper Lorentz group into irreducible representations. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 872–875. (Russian)

Let G denote the homogeneous Lorentz group and let L and M be two members of the principal series of irreducible unitary representations of G. The author announces a theorem giving an explicit decomposition of the tensor product $L\otimes M$ as a direct integral of members of this same principal series. Such a decomposition was given earlier by the reviewer. In Ann. of Math. (2) **55** (1952), 101–139 [MR **13**, 434] it was shown that the tensor product in question can be represented in a certain way as an induced representation, and in ibid. (2) **58** (1953), 193–211 [MR **15**, 101] this induced representation was analyzed into irreducible parts. The author's results are formulated in quite a different manner from those of the reviewer. He is apparently not aware of this earlier treatment of the problem.

{Added in proof: In a letter to the reviewer the author asserts that his results go beyond the reviewer's in including "a Plancherel formula". The reviewer can only guess what the author means by a Plancherel formula in this context. Hewever, his very explicit and computational statement has features which can be interpreted as implying such.} *G. W. Mackey* (Cambridge, Mass.)

7229:

Naĭmark, M. A. On the resolution of irreducible representations of the principal series of a complex unimodular group of order n into representations of a second order complex unimodular group. Dokl. Akad. Nauk SSSR **121** (1958), 590–593. (Russian)

Let A_n denote the group of all $n\times n$ unimodular complex matrices. Let H denote the subgroup of all members of A_n which agree with the identity matrix outside of the 2×2 upper-left-hand corner. H is clearly isomorphic to A_2. Let T denote a member of the principal series of irreducible unitary representations of A_n. Let $\hat{T}$ denote its restriction to H so that $\hat{T}$ may be regarded as a representation of A_2. In this note the author gives an explicit integral decomposition of $\hat{T}$ into irreducibles in the special case in which n is equal to three. He indicates that the same methods may be applied for any n. Only representations in the principal series of A_2 occur in the decomposition. *G. W. Mackey* (Cambridge, Mass.)

7230:

de Foglio, Susana Fernández Long. Extension de la différentielle d'Hadamard-Fréchet aux applications entre deux espaces vectoriels L. C. R. Acad. Sci. Paris **248** (1959), 1108–1110.

By a vector space of type L is meant here a vector space in which there is provided a notion of limit of a sequence satisfying axioms as given in Kuratowski's Topologie, vol. I [4th ed., Państwowe Wydawnictwo Naukowe, Warsaw, 1958; MR **19**, 873], § 14, and such that the vector space operations are continuous in the sequential sense. This paper is concerned with the definition and a few elementary propositions about differentials in the context of such spaces. No proofs are given. The derivative of a vector-valued function is defined in the obvious natural way. A function f from one vector space E of type L to another such space F is called differentiable at x_0 if there exists a continuous linear mapping U of E into F such that, whenever $g(t)$ has a derivative at t_0, with $g(t_0)=x_0$, then $f[g(t)]$ has the derivative $U[g'(t_0)]$ at t_0. The author states that f can fail to be continuous at x_0, even though differentiable. This anomaly cannot occur, however, if E is a true topological linear space each point of which has a countable fundamental set of neighborhoods. If to the definition of f being differentiable at x_0 is added the requirement that f is continuous at x_0, then the chain rule for differentials can be proved. There is also a theorem for differentials and partial differentials of functions of several variables. *A. E. Taylor* (Los Angeles, Calif.)

7231:

Shimogaki, Tetsuya. A generalization of Vainberg's theorem. I. Proc. Japan Acad. **34** (1958), 518–523.

If E is a measurable subset of the Euclidean n-space and $f(u, t)$ a real-valued function defined for all real u and $t\in E$, such that it is continuous in u and measurable in t for all u, then $x(t)\rightarrow f(x(t), t)$, where $x(t)$ is a measurable function on E, is a mapping of the space of all measurable functions on E into itself. Vainberg [Dokl. Akad. Nauk SSSR **92** (1953), 213–216; MR **15**, 439] showed that in order that $x\rightarrow f(x(t), t)$ map L_p into $L_{p_1}(p, p_1>0)$ it is necessary and sufficient that there exist a positive number γ and a function $a(t)\in L_{p_1}$ such that $|f(u, t)|\leqq a(t)+\gamma|u|^{p/p_1}$ for all $t\in E$, $-\infty<u<\infty$.

Let R be a modulared semi-ordered linear space. The author calls a mapping H of R into itself splittable if $[N]H(x)=H([N]x)$ for all $x\in R$ and $N\subset R$, where $[N]$ is the projection operator defined by the least normal manifold containing N. (Note that the mapping $x(t)\rightarrow f(x(t), t)-f(0, t)$ is splittable.) He characterizes those modulared semi-ordered linear spaces R on which every splittable operator H satisfies a Vainberg inequality: $|H(x)|\leqq c+\gamma|x|$, where $\gamma>0$, $c, x\in R$ and $c>0$. Further details and explanations will be given in another paper. *W. A. J. Luxemburg* (Pasadena, Calif.)

7232:

Altman, M. On the approximate solutions of functional equations in L^p spaces. Colloq. Math. **6** (1958), 127–134.

In an earlier paper [Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 461–465; MR **19**, 984] the author suggested

solving the equation $F(x)=0$, where F is a real-valued Fréchet differentiable function on a Banach space X, by means of the iteration

$$x_{n+1}=x_n-\frac{F(x_n)}{F'(x_n)(y_n)}y_n \quad (n=0, 1, 2, \cdots),$$

provided it is possible to choose y_n in X so that $\|y_n\|=1$ and $F'(x_n)(y_n)=\|F'(x_n)\|$. In this note he shows how the y_n can be chosen in the case that $X=L^p$, $p>1$. Application is also made to the solution of $P(x)=0$, where P maps L^p into L^p, by studying the function F given by $F(x)=\|P(x)\|$. [See also M. Altman, ibid. **5** (1957), 1031–1036, 1099–1103; MR **19**, 984, 1068.]

R. G. Bartle (Urbana, Ill.)

7233:

Suzuki, Noboru. A linear representation of a countably infinite group. Proc. Japan Acad. **34** (1958), 575–579.

The author proves that every countable group is isomorphic to a group of outer automorphisms (i.e. a group of *-automorphisms with only the identity element inner) of an approximately finite factor on a separable Hilbert space. This result is deduced, by means of a complicated construction, from theorems of Murray and von Neumann [Ann. of Math. (2) **44** (1943), 716–808; MR **5**, 101] and Dixmier [ibid. **59** (1954), 279–286; MR **15**, 539].

D. A. Edwards (Oxford)

CALCULUS OF VARIATIONS

See also 7101.

7234:

Bellman, Richard. On a differential inequality of Cesari and Turner. Rend. Circ. Mat. Palermo (2) **7** (1958), 34–36.

The author gives an alternative proof of an integral inequality of Cesari and Turner [same Rend. **6** (1957), 109–113; MR **20** #2409] and suggests some variants and extensions. The original inequality is of importance as a tool in surface area.

L. C. Young (Madison, Wis.)

7235:

Nikol'skiĭ, S. M. A variational problem of Hilbert. Izv. Akad. Nauk SSSR. Ser. Mat. **22** (1958), 599–630. (Russian)

Let Λ be a closed manifold in R_n and let a, b, c, denote functions of positions on Λ. The author studies, under certain smoothness assumptions, the minimum of the Dirichlet integral over R_n of a function f continuous except on Λ where it has limits A, B from inside and from outside subject to $aA-bB=c$.

L. C. Young (Madison, Wis.)

GEOMETRIES, EUCLIDEAN AND OTHER

7236:

Karzel, Helmut. Zentrumsgeometrien und elliptische Lotkerngeometrien. Arch. Math. **9** (1958), 455–464.

As in his previous investigations the author considers a group $\mathfrak{G}$ generated by a set $\mathfrak{E}$ of involutoric elements (corresponding to lines in a plane). A point P_{ab} or pencil of lines consists of the x with abx inv (i.e., abx is involutoric), where $a\neq b$, a, $b\in\mathfrak{E}$. The pencil P_{ab} is proper if $P_{ab}\cap P_{cd}\neq 0$ for any P_{cd}. Starting from the axioms: (I) If $a\neq b$ and abx_i inv, $i=1, 2, 3$, then $x_1x_2x_3\in\mathfrak{E}$; (II) every line carries at least two proper points; one can define reflections in lines, obtaining a geometry $(\mathfrak{G}^*, \mathfrak{E}^*)$, which satisfies (I) only if the center of $\mathfrak{G}$ is trivial. The group $\mathfrak{G}^*$ is discussed. Then (I) is weakened to (I)*: If $a\neq b$ and abx_i inv, $i=1, 2, 3$, then either $x_1x_2x_3\in\mathfrak{E}$ or $x_1x_2x_3=1$. The $(\mathfrak{G}^*, \mathfrak{E}^*)$ satisfy (I)* and (II). The principal purpose of the paper is the enumeration of all geometries satisfying (I)* and (II), which fall into five categories.

H. Busemann (Cambridge, Mass.)

7237:

Tekse, Kálmán. Über mit Kreislineal durchführbare Konstruktionen. Mat. Lapok **7** (1956), 255–261. (Hungarian. Russian and German summaries)

Das Problem der Konstruktion mit Hilfe eines Kreislineals wurde zuerst von Hjelmslev [Mat. Tidsskr. **1938**, 77–85] aufgeworfen. Unter einem Kreislineal versteht man eine Kreisplatte (Münze) vom Radius R, mit Hilfe derer man einen Kreis vom Radius R zeichnen, und auf den Umfang des so erhaltenen Kreises gegebene Bögen auftragen kann. Mit dem Kreislineal ist auch die Konstruktion eines Kreises gestattet, der einen gegebenen Kreis in einem bestimmten Punkt berührt. Hjelmslev hat gezeigt, dass nur mit einem Kreislineal, wenn auch nicht alle, sodoch eine Reihe von quadratischen (mit Zirkel und Lineal durchführbaren) Grundkonstruktionen ausgeführt werden können. Verf. beweist folgenden Satz: Jede quadratische Konstruktion in der Ebene ist mit Hilfe eines Kreislineals vom Radius R ausführbar, falls 2 Strahlenbüschel in der Ebene gegeben sind, deren Träger den Abstand $2R$ haben.

L. Gyarmathi (Debrecen)

7238:

Drs, Ladislav. Über die zentrale Axonometrie. Časopis Pěst. Mat. **83** (1958), 330–335. (Czech. Russian and German summaries)

Continuing earlier work [same Časopis **82** (1957), 165–174; MR **19**, 572] the author proves theorems useful for the constructions of those plane configurations O, J_i, U_i ($i=1, 2, 3$) which are obtained by central projection of a rectangular coordinate system in space with origin O', unit points J_i' and points at infinity U_i' on the axes $O'J_i'$.

F. A. Behrend (Melbourne)

7239:

Bonfiglioli, Luisa. Sezione di un prisma e di una piramide secondo un triangolo dato. Riveon Lematematika **12** (1958), 3–12. (Hebrew. Italian summary)

L'a., nel presente articolo, dà un metodo grafico, empirico, per risolvere il problema di segare una piramide triangolare secondo un triangolo uguale a un triangolo dato, e un prisma triangolare secondo un triangolo simile a un triangolo dato.

M. Piazzolla-Beloch (Ferrara)

7240:

Mandan, Sahib Ram. An S-configuration in Euclidean and elliptic n-space. Canad. J. Math. **10** (1958), 489–501.

In general, $n+1$ spheres (i.e., hyperspheres) in Euclidean n-space have $n(n+1)$ centers of similitude, forming a so-called S-configuration. When the centers of the spheres (of radii r_i) are used to form the simplex of reference, the centers of similitude of the first two have the barycentric coordinates $(r_2, \pm r_1, 0, \cdots, 0)$. Midway between these two is the point $(r_2^2, -r_1^2, 0, \cdots, 0)$. The $\frac{1}{2}n(n+1)$ such midpoints lie on the so-called Newtonian

hyperplane $\sum r_i^2 x_i=0$. For a projective treatment of the S-configuration it is convenient to make a change of scale so that the $n(n+1)$ points are simply the permutations of $(1, \pm 1, 0, \cdots, 0)$. Then the hyperplane $\sum x_i=0$ contains a sub-configuration of $\frac{1}{2}n(n+1)$ points, and there are 2^n such sub-configurations given by changing signs in the equation. The $n(n+1)$ points are recognized as sections of the diagonals joining pairs of opposite vertices, such as $(\pm 1, \pm 1, 0, \cdots, 0)$, of the $(n+1)$-dimensional polytope $t_1\beta_{n+1}$ [the D_n of Coxeter, same J. **3** (1951), 391–441; MR **13**, 443; see pp. 405, 414].

{It is unfortunate that the "graphical symbols" were printed with nodes that are too small to be seen, and that the one on the last page lacks the mark 4 under its final branch.} *H. S. M. Coxeter* (Toronto, Ont.)

7241:

Schneider, Z.; und Stankovitsch, B. Über die Anzahl und Anordnung der Diagonalschnitte in einem regelmässigen n-Eck. Elem. Math. **14** (1959), 6–11.

The greatest possible number of intersections of diagonals of a convex n-gon is $\binom{n}{4}$ [Coxeter, *The real projective plane*, University Press, Cambridge, 1955; MR **16**, 1143; p. 133, Ex. 7]. When the n-gon is regular, the number may be smaller because the diagonals may concur in sets of three or more [Coxeter, Amer. J. Math. **62** (1940), 457–486; MR **2**, 10; p. 463]. The authors ignore this reduction, i.e., they regard k concurrent diagonals as intersecting $\binom{k}{2}$ times. When n is even, they might be expected to count the center $\binom{\frac{1}{2}n}{2}$ times, but actually they prefer not to count it at all.

H. S. M. Coxeter (Toronto, Ont.)

7242:

Reeve, J. E. A further note on the volume of lattice polyhedra. J. London Math. Soc. **34** (1959), 57–62.

The author has slightly strengthened the results of his earlier papers [Proc. London Math. Soc. (3) **7** (1957), 378–395; C. R. Acad. Sci. Paris **246** (1958), 2989–2991; MR **20** #1954, #1955] and has related his four-dimensional conjecture to a paper on hypersolid angles by D. M. Y. Sommerville [Proc. Roy. Soc. London. Ser. A **115** (1927), 103–119; cf. W. Höhn, *Winkel und Winkelsumme im n-dimensionalen euklidischen Simplex*, Thesis, Eidgenössische Technische Hochschule, Zürich, 1953; MR **15**, 55].

H. S. M. Coxeter (Toronto, Ont.)

7243:

Court, N. A. Desmic systems of tetrahedrons. Amer. Math. Monthly **66** (1959), 123–125.

Se due tetraedri sono ciascuno armonico ad un terzo, le rette che da un vertice di questo proiettano gli otto vertici dei primi due appartengono ad un cono quadrico. Se un tetraedro appartiene a due diversi sistemi desmici, i 16 vertici dei rimanenti 4 tetraedri appartengono ad una quartica di prima specie. *D. Gallarati* (Genoa)

7244:

Dubikajtis, L. On the order of points and hyperplanes in n-dimensional projective geometry. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 607–610.

Four distinct points A, B, C, D of a (real) projective line are said to form a cycle $\{A, C, B, D\}$ if A and B separate C and D. This notion is fundamental in developing order relations on the line. The author extends the notion to n-dimensional projective space in the following way. First, the notion of cycle can be dualised in an obvious way to express a relation between four hyperplanes of a pencil. Now let A_i, for $1 \leq i \leq n+3$, be $n+3$ points in P_n, no $n+1$ of which lie in a hyperplane. We say that $\{A_1, A_2, \cdots, A_{n+3}\}$ is a cycle if, for every set of integers p, q, r, s such that $1 \leq p < q < r < s \leq n+3$, the four hyperplanes $\{\alpha_p, \alpha_q, \alpha_r, \alpha_s\}$ form a cycle, where α_p is the hyperplane containing all the points except A_q, A_r, A_s, and $\alpha_q, \alpha_r, \alpha_s$ are defined similarly. The property is unaltered if the points are permuted in cyclic order, or if the order of the points is reversed; moreover, if $n+3$ points in P_n are given, no n of which lie in a hyperplane, they determine a unique cycle. The dual notion can also be defined. The author remarks that, when $n=2$, this notion can easily be described in geometric terms; namely, five lines in a plane, no three of which are concurrent, divide the plane into eleven regions, exactly one of which is bounded by a pentagon. The sides of this pentagon, taken in order, define the cycle. *J. A. Todd* (Cambridge, England)

7245:

Rosati, Luigi Antonio. I gruppi di collineazioni dei piani di Hughes. Boll. Un. Mat. Ital. (3) **13** (1958), 505–513. (English summary)

In this paper, the determination of the collineation groups of the "Hughes planes" [D. R. Hughes, Canad. J. Math. **9** (1957), 378–388; MR **19**, 444] is finished. The planes are the only finite non-Veblen-Wedderburn projective planes known at the present time. Zappa has determined [Boll. Un. Mat. Ital. (3) **12** (1957), 507–516; MR **19**, 876] that the group contains the product of the projective group of a certain Desarguesian subplane with the automorphism group of the near-field which underlies the construction; here it is shown that this is indeed the entire group. Further interesting results on the transitive constituents of points and lines of a Hughes plane are included. *D. R. Hughes* (Chicago, Ill.)

CONVEX SETS AND DISTANCE GEOMETRIES

See also 7033, 7202, 7213, 7593.

7246:

Molnár, J. Über Sternpolygone. Publ. Math. Debrecen **5** (1958), 241–245.

In 1946 Krasnoselskiĭ [Mat. Sb. (N.S.) **19(61)** (1946), 309–310; MR **8**, 525] proved that if T is a compact set in E_n such that each $n+1$ boundary points of T are "visible" from some point of T, then T is star-shaped. The present author proves a generalization for polygons in the plane. A side XY is called a side of inflection if the angle at one end is concave while the angle at the other end is convex. Let P be a plane polygon with at least three sides of inflection. If to each three sides of inflection a point of P can be found from which at least one point of each of the three sides can be seen, then P is star-shaped.

A. L. Shields (Ann Arbor, Mich.)

7247:

Rogers, C. A. Lattice coverings of space with convex bodies. J. London Math. Soc. **33** (1958), 208–212.

It is shown that there exists a lattice covering of n-dimensional space by the translates of an arbitrary convex body with density 2^n. The main tool is an estimate of the average density of the points not covered by the lattice translates of a fixed body of volume V with

$0 < V \leqq 1$ where the average is extended over all lattices of determinant 1. With the help of Siegel's mean value theorem it is shown that this average is no greater than $1 - V + \frac{1}{2}V^2$.

A sharpening of the density result to $(1.8774)^n$ for large n is announced for a later paper.

E. G. Straus (Los Angeles, Calif.)

7248:

Ohshio, Shigeru. Parallel series to a closed convex curve and surface and the differentiability of their quantities. Sci. Rep. Kanazawa Univ. **6** (1958), 15–24.

The author defines parallel figures of a convex polygon which do not agree with those generally so termed and are not necessarily convex or even simple, and obtains for their signed areas and signed boundary lengths the usual Steiner formulae. He states without proof extensions to convex curves and convex surfaces. Such extensions could perhaps apply also, to some extent, irrespective to convexity. *L. C. Young* (Madison, Wis.)

GENERAL TOPOLOGY, POINT SET THEORY

See also 7202.

7249:

Mycielski, Jan. On the decomposition of a segment into congruent sets and related problems. Colloq. Math. **5** (1957), 24–27.

Viene generalizzata una nota proposizione di J. von Neumann [Fund. Math. **11** (1928), 230–238], dimostrando il seguente teor., molto interessante sia per la teoria generale delle potenze degli insiemi, sia per la teoria euclidea delle congruenze: se $\aleph_0 \leqq m \leqq 2^{\aleph_0}$, allora ognuno degli intervalli unitari $(0<x<1)$, $(0 \leqq x < 1)$, $(0 \leqq x \leqq 1)$ è la somma di m insiemi disgiunti, congruenti l'uno all'altro per traslazione, ciascuno avente potenza $2^{\aleph_0}$. Alcune osservazioni critiche concludono questo lavoro.

T. Viola (Torino)

7250:

Rosen, Ronald H. On tree-like continua and irreducibility. Duke Math. J. **26** (1959), 113–122.

The main result is that a (decomposable) tree-like continuum having a basis of treelike coverings with k or fewer end elements is irreducible about some set of k points. There are further technical results on decompositions and on subcontinua of treelike continua; for example, a monotone upper semi-continuous decomposition yields another tree-like continuum.

J. Isbell (Seattle, Wash.)

7251:

Bing, R. H.; and Jones, F. B. Another homogeneous plane continuum. Trans. Amer. Math. Soc. **90** (1959), 171–192.

This paper describes a homogeneous plane continuum that is neither a simple closed curve nor a pseudo-arc. The example, which is a circle of pseudo-arcs, was discovered independently by the two authors in 1954. The paper contains an instructive history of the problem of discovering homogeneous plane continua, and an excellent bibliography. *D. W. Hall* (Endicott, N.Y.)

7252:

Picone, Mauro. Sul concetto di limite. Ann. Mat. Pura Appl. (4) **46** (1958), 349–367. (English summary)

This paper deals with certain types of limit points associated with families of sets in a Hausdorff space. A theorem is proved on the invariance of these limit points under continuous transformations. Some properties connected with the total variation of a function whose values lie in a metric space are derived. In discussing the special case of the real line, the author adjoins three ideal points, ∞, $+\infty$, and $-\infty$. The neighborhoods assigned to these points do not satisfy the Hausdorff separation axiom γ.

L. M. Graves (Chicago, Ill.)

7253:

Roppert, Josef. Untersuchungen über Fréchet-Räume. Monatsh. Math. **62** (1958), 345–356.

Von den in Fréchets Buch [*Les espaces abstraits*, Gauthier-Villars, Paris, 1928] definierten Begriffen ausgehend untersucht der Verf. zwei Mengenoperationen $\bar{E}$, $H(E)$, die auf dem System aller Teilmengen eines (V)-Raumes K definiert sind. $\bar{E}$ (Hülle) ist der Durchschnitt aller abgeschlossenen Mengen, die E enthalten; $H(E)$ ist die Summe der Menge E und ihrer Ableitung. Für eine Folge $\{A_n\}$ der Teilmengen von K wird die Menge $\limsup A_n$ ähnlich wie in Kuratowskis Buch [Topologie I, 2. Aufl., Monografie Matematyczne, t. 20, Warszawa-Wrocław, 1948; MR **10**, 389; S. 243] definiert. Unter einer additiven Operation f wird eine solche verstanden, für die $f(A+B)=f(A)+f(B)$ bei jedem A, $B \subset K$ ist. Es wird bewiesen, daß die Gültigkeit der Gleichung $\limsup A_n + \limsup B_n = \limsup(A_n+B_n)$ mit der Additivität der Operation H äquivalent ist; aus der letzteren folgt die Additivität der Hülle, die mit der Gültigkeit der Bedingung $\overline{A-B} \supset \bar{A}-\bar{B}$ äquivalent ist. Aus der Additivität der Hülle folgt die Additivität einer anderen Operation, die durch nacheinandergehende Hüllen- und Komplementbildung entsteht. Aus der gegebenen Menge $E \subset K$ können durch Hüllen- und Komplementbildung höchstens 14 verschiedene Mengen abgeleitet werden, womit ein Resultat von Kuratowski [loc. cit. S. 24] verallgemeinert wird. *M. Novotný* (Brno)

7254:

Novák, Josef. Über die eindeutigen stetigen Erweiterungen stetiger Funktionen. Czechoslovak Math. J. **8** (83) (1958), 344–355. (Russian summary)

Let X be a Fréchet L-space for which also the Urysohn axiom holds, namely: if $\{x_n\}$ does not converge to x, then there exists a subsequence $\{x_{n_k}\}$ such that no further subsequence $\{x_{n_{k_i}}\}$ converges to x. The author calls "Abschliessung" ("closure") vA of a set $A (\subset X)$ the set of all limit points $x = \lim x_n$ with $x_n \in A$. By transfinite induction the "αth Abschliessung" $v^\alpha A$ of the set A is also defined for each countable ordinal number α. Moreover, he sets $uA = \bigcup v^\alpha A$ where the union refers to all countable ordinal numbers α. The author then gives conditions under which a continuous real function f defined on A can be extended to a continuous function on vA (or on uA) in a unique manner. Then he applies his results to the extension of a bounded measure on a ring of sets to the corresponding σ-ring of sets without using the Carathéodory outer measure. *A. Rosenthal* (Lafayette, Ind.)

7255:

Császár, A.; et Mrówka, S. Sur la compactification des espaces de proximité. Fund. Math. **46** (1959), 195–207.

The authors define a proximity basis as a family $\mathfrak{B}$ of subsets such that for any two distant sets, A, B, there are distant sets U, V, in $\mathfrak{B}$ with $A \subset U$, $B \subset V$. They show that this is essentially the same as a basis for the natural compactification [Smirnov, Mat. Sb. N.S. **31(73)** (1952), 543–574; MR **14**, 1107] (passing up by interiors of closures, down by relativization), and add some remarks about the cardinality of bases. *J. Isbell* (Seattle, Wash.)

7256:

Whyburn, G. T. On convergence of mappings. Colloq. Math. **6** (1958), 311–318.

In this paper the author develops conditions sufficient that sequences of compact mappings and that sequences of quasi-open mappings converge almost uniformly. An example of a sequence of homeomorphisms of the plane onto itself which converges to the identity but for which the convergence fails to be almost uniform is first given. This illustrates that even the strongest conditions on the individual mappings are not sufficient for almost uniform convergence.

The first theorem is concerned with a sequence of compact monotone mappings $f_n(X)=Y_n$ and a limit mapping $f(X)=Y'$ where X, Y and Y' are closed generalized continua in a locally compact separable metric space with $Y'\subset Y$ and for each $y\in Y$, Y' intersects each component of $Y-y$. Under the conditions that: (a) the sets Y_n converge 0-regularly to Y (i.e., for any $\varepsilon>0$ there is an N and a $\delta>0$ such that for all $n>N$ any pair of points $y, y'\in Y_n$, of distance $<\delta$, are in a connected set in Y_n of diameter $<\varepsilon$); and (b) for each $x\in X$ and $\varepsilon>0$ there exists an ε-neighborhood U of x with boundary C such that $\lim\sup f_n(C)$ is strictly contained in $f(C)$ (i.e., for every $\varepsilon>0$ almost all of the $f_n(C)$ are in the ε-neighborhood of $f(C)$); it is shown that the sequence $\{f_n(x)\}$ converges almost uniformly to $f(x)$ on X and also, if each $f^{-1}(y)$, $y\in Y'$, has a non-empty compact component, that f is compact and monotone.

The second theorem is concerned with a sequence of quasi-open mappings $f_n:X\to Y$, with X and Y locally connected generalized continua such that, for each $y\in Y$, each component of $Y-y$ is non-compact. Under the condition that there exists a mapping $f:X\to Y$ such that for each $x\in X$ and $\varepsilon>0$ there is a conditionally compact region R, boundary C, $x\in R$ and R in the ε-neighborhood of $f^{-1}(f(x))$ with $\lim\sup f_n(C)$ strictly contained in $f(C)$, it is shown that the sequence $\{f_n(x)\}$ converges almost uniformly to $f(x)$ on X.

C. J. Titus (Ann Arbor, Mich.)

7257:

Marr, J. M. On essential fixed points. Proc. Amer. Math. Soc. **10** (1959), 148.

Theorem: If X is a compact Hausdorff space with the fixed point property, then there is an $f\in X^X$ (compact open topology) such that each fixed point p of f is essential, i.e., for each neighborhood U of p there is a neighborhood N of f such that every $g\in N$ has a fixed point in U. Proof: set $f(X)=x_0\in X$.

7258:

Lelek, A. Sur les involutions multivalentes. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 511–517.

This paper contains a generalization of a lemma of Kuratowski concerning continuous involutions defined on locally connected and unicoherent continua. [See I. Bernstein, Fund. Math. **43** (1956), 89–94; MR **18**, 56; in particular, p. 90; and K. Zarankiewicz, Bull. Acad. Polon. Sci. Cl. III **2** (1954), 117–120; MR **16**, 502; in particular, p. 117.]

Let $F:X\to 2^X$ be a mapping. Then a subset A of X has property (α) if and only if there exists a composante C of the set $X-A$ such that $F(A)$ is contained in C. Theorem 1: Let the space X be a locally connected and unicoherent continuum, the function $F:X\to 2^X$ upper semicontinuous, and G an open subset of X. Then G, or better $X-G$, has a composante not possessing property (α).

A point x is called an invariant point of the function $F:X\to 2^X$ provided $x\in F(x)$. More generally, a subset A of X is said to be a partially invariant subset of this function provided A and $F(A)$ have a non-void intersection. The symbol $C(X)$ denotes the set of all non-void subcontinua of X.

Lemma: If the function $F:X\to C(X)$ is upper semicontinuous and the subset A of X is connected, then $F(A)$ is connected. Theorem 2: Let X be a locally connected and unicoherent continuum, let $F:X\to C(X)$ be upper semicontinuous, and let G be an open subset of X. Then G, or better $X-G$, has a composante which is a partially invariant subset of the function F.

We call an upper semicontinuous function $\Phi:X\to 2^X$ a multivalent involution when $x_1\in\Phi(x_2)$ for every $x_2\in\Phi(x_1)$. We say that an upper semicontinuous function $\Phi:X\to 2^X$ is a multivalent involution in the large sense when $x_1\in\Phi(x_2)$ for at least one $x_2\in\Phi(x_1)$. Theorem 3: Let X be a locally connected and unicoherent continuum, let the function $\psi:X\to C(X)$ be a multivalent involution in the large sense, and let the open subset G of X satisfy the two conditions $\psi(G)\subset G$ and $\psi(X-G)\subset X-G$. Then G, or better $X-G$, has a composante A such that $\psi(A)=A$. Theorem 4: Let X be a locally connected and unicoherent continuum, let $\Phi:X\to C(X)$ be a multivalent involution, and let G be an open subset of X satisfying $\Phi(G)\subset G$. Then G, or better $X-G$, has a composante A such that $\Phi(A)=A$.

Theorem 4 is the desired generalization of the lemma of Kuratowski.

D. W. Hall (Endicott, N.Y.)

7259:

Kelley, J. L. On mappings of plane sets. Colloq. Math. **6** (1958), 153–154.

Two conjectures are made which would settle two old unsolved problems. The problems are: 1) Does there exist a fixed point free continuous mapping of a plane non separating continuum into itself? 2) Does there exist a continuous fixed point free periodic map of a connected set, which fails to cut the plane, into itself? The conjectures are: Conjecture A: If f is a continuous fixed point free periodic map of a connected plane set into itself, then the map $z\to f(z)-z$ is essential. Conjecture B: If f is a continuous fixed point free mapping of the plane continuum X into itself, then the map $z\to f(z)-z$ is essential. The author proves conjecture A in case the map is of period 2, and proves conjecture B in case X is a neighborhood retract of the union of X and the unbounded component of the complement of X.

Haskell Cohen (Baton Rouge, La.)

7260:

Kinoshita, S. On orbits of homeomorphisms. Colloq. Math. **6** (1958), 49–53.

Let X be a compact metric space and let f be a continuous mapping of X into X. Let $P_+(f)$ be the set of points of X each of which has the property that its positive semi-orbit closure under f is dense in X. Let $Q_+(f)=X-P_+(f)$. It is shown that if $P_+(f)\neq 0$, then $P_+(f)$ is a dense G_δ set. If $Q_+(f)\neq 0$, then $Q_+(f)$ is a dense F_σ set.

W. R. Utz (Columbia, Mo.)

ALGEBRAIC TOPOLOGY

See also 7302, 7303.

7261:

James, I. M.; and Whitehead, J. H. C. Homology with zero coefficients. Quart. J. Math. Oxford Ser. (2) **9** (1958), 317–320.

The authors give some examples of non-trivial homology theories with zero coefficient group, and also give a general method for constructing such theories [a 'homology theory' is one which satisfies axioms 1–7 in chapter I of Eilenberg and Steenrod, *Foundations of algebraic topology*, Princeton Univ. Press, 1952; MR **14**, 398].

D. Buchsbaum (Providence, R.I.)

7262:

Berstein, Israel. Essential and inessential complexes. Comment. Math. Helv. 33 (1959), 70–80.

Let K be an n-dimensional homogeneous CW complex and τ one of its n-cells; let $L=K-\tau$. K is called cyclic [cf. H. Hopf and E. Pannwitz, Math. Ann. **108** (1933), 433–465] if the injection

$$H_n(K, G)\to H_n(K, L; G)$$

is nontrivial for one of G=the integers or the integers mod m, some m. The author calls K co-cyclic if

$$H^n(K, L; A)\to H^n(K, A)$$

is nontrivial for some coefficients A (possibly local).

K is called essential if it cannot be deformed into a proper subset. For $n\geqq 3$ Hopf and Pannwitz had shown that "essential" and "cyclic" are equivalent for simply connected K. The author shows that, again for $n\geqq 3$, "co-cyclic" is also equivalent to "essential", without the restriction to simple connectivity.

Answering a question left open by Hopf and Pannwitz, the author shows: There exists a finite homogeneous 2-dimensional complex which is simply connected and essential but not cyclic. *V. Gugenheim* (Baltimore, Md.)

7263:

Toda, Hirosi. On exact sequences in Steenrod algebra mod. 2. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **31** (1958), 33–64.

Le calcul de la composante 2-primaire des groupes d'homotopie stables des sphères au moyen de la méthode des fibrations de Cartan-Serre, ou de toute autre méthode équivalente, nécessite une étude détaillée des relations qui existent dans l'algèbre de Steenrod $A^*=\sum_{n\geq 0}A^n$ des opérations cohomologiques stables à coefficients dans Z_2. L'Auteur établit ici les résultats suivants (on note φ_t, resp. φ_t^*, l'application $A^*\to A^*$ définie par la multiplication à droite, resp. à gauche, par Sq^t; on note aussi φ_t toute application déduite de φ_t par passage aux quotients).

Théorème 1: On a les suites exactes (dont les 4 premières forment un cycle fermé):

(1) $A^* \xrightarrow{\varphi_2} A^* \xrightarrow{\varphi_2} A^*/\varphi_1A^*,$

(2) $A^* \xrightarrow{\varphi_2} A^*/\varphi_1A^* \xrightarrow{\varphi_2} A^*/\varphi_1A^*,$

(3) $A^*/\varphi_1A^* \xrightarrow{\varphi} A^*/\varphi_1A^* \xrightarrow{\varphi_2} A^*,$

(4) $A^*/\varphi_1A^* \xrightarrow{\varphi_2} A^* \xrightarrow{\varphi_2} A^*,$

(5) $A^*/\varphi_1A^* \xrightarrow{\varphi_3} A^*/\varphi_1A^* \xrightarrow{\varphi_2} A^*/\varphi_1A^*.$

Théorème 2: Soit $B_{(k)}=\sum_{i\geq 0}B_{(k)}{}^i$ le noyau-image de la suite exacte (k) du théorème 1 ($k=1, \cdots, 5$). Soit $H_{(k)}{}^i$ le groupe d'homologie de la suite

$$B_{(k)}{}^{i-1} \xrightarrow{\varphi_1^*} B_{(k)}{}^i \xrightarrow{\varphi_1^*} B_{(k)}{}^{i+1};$$

on a $H_{(k)}{}^i=0$ ou Z_2. Pour que $H_{(k)}{}^i=Z_2$, il faut et il suffit que i soit $\geqq 2$ et satisfasse aux conditions: $i\equiv 0$ (4) pour $k=1$; $i\equiv 1$ (4) pour $k=2$ ou 3; $i\equiv 3$ (4) pour $k=4$; $i\equiv 1$ (2) pour $k=5$.

On a d'autres résultats, tels que celui-ci: la suite

$$A^{i-8} \xrightarrow{\varphi} A^{i-4}/\varphi_1A^{i-5} \xrightarrow{\varphi_4} A^i/(\varphi_1A^{i-1}+\varphi_2A^{i-2})$$

est exacte pour $i<22$; il est conjecturé que ceci est vrai pour tout i.

Ces résultats, et une étude des opérations de Bockstein itérées (non partout définies) dans la cohomologie d'un fibré, permettent à l'Auteur de calculer à nouveau les composantes 2-primaires des groupes stables $\pi_{n+k}(S_n)$, pour $k\leqq 13$. Cette méthode ne permet pas encore de lever certaines indéterminations, ce qui laisse subsister une ambiguïté pour $k=14$. *H. Cartan* (Paris)

7264:

Inoue, Yoshiro. On cohomology operations of the second kind. J. Math. Soc. Japan **10** (1958), 249–254.

The cohomology operations θ_2 studied by the author satisfy the following conditions. θ_2 is defined on Ker θ_1, where θ_1: $H^n(K, L; A)\to H^p(K, L; B)$ is an operation of the first kind.

The values of θ_2 lie in a natural quotient group of $H^q(K, L; C)$. The author's main tool is a "canonical fibration", neatly constructed in CSS-theory. His main purpose is to prove that the operations he studies admit universal examples. This proposition, if correctly interpreted, is not in doubt; unfortunately, the accuracy of the details as given is impaired by an error in lemma 4. This lemma states the equality of two indeterminacy subgroups $G_y(K, L)$ and $G_z(K, L)$; it is proved that $G_y(K, L)\supset G_z(K, L)$, but the converse inclusion does not hold in general, as can be shown by counter-examples.

{In a letter to the reviewer, the author indicates that the paper may be corrected as follows. P.252: for the statement and proof of lemma 4, substitute "If $y, z\in H^q(X; C)$ are equivalent, then $G_y(K, L)=G_z(K, L)$". P.252, lines 34, 35: for the words "is called to be defined by y or $[y]$" substitute "is called to be regular". P.252, line 36 to P.253, line 7: delete this paragraph. P.253, line 9: for "minimal" substitute "regular". P.253, line 25: insert a full stop after the words "the proof is complete", and delete the remainder of the paper.}

J. F. Adams (Cambridge, England)

7265:

Dugundji, J. Cohomology of equivariant maps. Trans. Amer. Math. Soc. **89** (1958), 408–420.

Let the group W operate on a complex K, let $h_n(K)$= integral n-chains mod boundaries. The natural map "integral n-chains$\to h_n(K)$" defines an element $e_W{}^n(K)$ in the equivariant cohomology group $H_W{}^n(K, h_n(K))$; this element has several "universal properties"; e.g.: For any W-group G, any element of $H_W{}^n(K, G)$ is the image of $e_W{}^n(K)$ under a W-homomorphism $h_n(K)\to G$; if $e_W{}^n(K)=0$, then $H_{n-1}(K)$ is free and $H^n(K, G)=H_n(K, G)=0$ for all G.

$e_W{}^n(K)$ is not invariant (if K is regarded as the triangulation of a space). This can be overcome by taking K as the total singular complex of a space; then, however, $H_W{}^n(K, h_n(K))$ and $e_W{}^n(K)$ are not "calculable"; a compromise is possible when $e_W{}^n(K)$ is in the image of the

natural homomorphism $H_W{}^n(K, H_n(K)) \to H_W{}^n(K, h_n(K))$ (note that $H_n(K) \subset h_n(K)$.) This image-group is invariant; $e_W{}^n(K)$ lies in it if $C_{n-1}(K)$ is W-free and if the identity $h_{n-1}(K) \to h_{n-1}(K)$ can be factored through a W-free group F. The author obtains a universal coefficient theorem for $H_W{}^n(K, G)$ (which, however, is expressed in terms of the groups $h_n(K)$ and $h_{n-1}(K)$), and relationships between the image of $e_W{}^n(K)$ under a map and certain obstruction cocycles.

V. Gugenheim (Baltimore, Md.)

7266:

James, I. M. Filtration of the homotopy groups of spheres. Quart. J. Math. Oxford Ser. (2) 9 (1958), 301–309.

Let A be a connected CW complex having a single vertex a_0, and let $\hat{A}$ denote the suspension of A. The author defines an increasing filtration on the homotopy group $\pi_{r+1}(\hat{A})$ as follows. Let A_∞ denote the "reduced product space of A", introduced previously by the author [Ann. of Math. (2) 62 (1955), 170–197; MR 17, 396]. The reduced product space is the free topological semi-group with a unit generated by the space A; let A_m denote the subspace of A_∞ consisting of those "words" of length $\leqq m$. Then A_m is a closed subspace of A_∞, and the sequence of spaces $\{A_m\}$ defines an increasing filtration on the space A_∞. This in turn defines an increasing filtration on the homotopy group $\pi_r(A_\infty)$. Since $\pi_r(A_\infty)$ is naturally isomorphic to $\pi_{r+1}(\hat{A})$ (because A_∞ has the same weak homotopy type as the space of loops on $\hat{A}$), there is defined a filtration on $\pi_{r+1}(\hat{A})$. This filtration is an invariant of A rather than $\hat{A}$.

The main interest lies in the case where A, and hence $\hat{A}$, is a sphere. In this case the author proves that if $\alpha \in \pi_{m+1}(S^{n+1})$ and $\beta \in \pi_{r+1}(S^{n+1})$, then the filtration of the composition $\alpha \circ \beta$ is $\leqq$ the product of the filtrations of α and β. In general, the inequality cannot be sharpened to an equality. An exception is the case where $m=2n$ and α is an element of Hopf invariant unity; then equality holds. The author also proves that the elements of order p in $\pi_{2p}(S^3)$ and $\pi_{4p-3}(S^3)$ have filtration $p-1$, and those of order p in $\pi_{4p-2}(S^3)$ have filtration p (for any odd prime p). Thus in the case of S^3 there exist elements of arbitrarily great filtration. Finally, the author computes the filtrations of all elements of $\pi_r(S^n)$ in case $r \leqq n+7$. Difficulties are encountered in the case of $\pi_{11}(S^3)$, and it appears that further analysis is required before this case can be dealt with.

{Reviewer's note: Recently J. F. Adams has defined a filtration on the stable homotopy groups of spheres which is quite different from that defined in the paper under review; see Comment. Math. Helv. 32 (1958), 180–214 [MR 20 #2711].} *W. S. Massey* (Providence, R.I.)

7267:

James, I. M. The intrinsic join: a study of the homotopy groups of Stiefel manifolds. Proc. London Math. Soc. (3) 8 (1958), 507–535.

Let F denote the field of real numbers, complex numbers, or quaternions, and let F_m denote the set of all m-tuples $(x_1, \cdots, x_m)$ of elements of F regarded as a right vector space over F. By a k-frame is meant a k-tuple of orthonormal vectors. The Stiefel manifold of all k-frames in F_m is denoted by $O_{m,k}$.

The main purpose of the present paper is to introduce a bilinear map $(\alpha, \beta) \to \alpha * \beta$ which assigns to each element $\alpha \in \pi_i(O_{m,k})$ and $\beta \in \pi_j(O_{n,k})$ an element $\alpha * \beta \in \pi_{i+j+1}(O_{m+n,k})$, called the "intrinsic join" of α and β. To define this operation, the author first defines a homeomorphic imbedding h of the join $O_{m,k} * O_{n,k}$ into $O_{m+n,k}$ as follows. Suppose $u=(u_1, \cdots, u_k) \in O_{m,k}$, $v=(v_1, \cdots, v_k) \in O_{n,k}$ and $0 \leqq t \leqq 1$. Define $h(u, v, t)=w$, where $w=(w_1, \cdots, w_k) \in O_{m+n,k}$ and $w_r=(u_r \cdot \cos \frac{1}{2}\pi t, v_r \cdot \sin \frac{1}{2}\pi t)$ for $1 \leqq r \leqq k$. The intrinsic join of α and β is the image of the ordinary join of α and β under the homomorphism h_* induced by h.

Let $p: O_{m,k} \to O_{m,l}$ be the map defined by $p(u_1, \cdots, u_k) = (u_1, \cdots, u_l)$, $1 \leqq l \leqq k$. Then p is a fibre map with $O_{m-l,k-l}$ as fibre. The author proves a couple of fundamental theorems which describe the behavior of the intrinsic join of α and β under the various homomorphisms of the exact homotopy sequence of this fibre space. Applications are promised in later papers [see the following review]. *W. S. Massey* (Providence, R.I.)

7268:

James, I. M. Cross-sections of Stiefel manifolds. Proc. London Math. Soc. (3) 8 (1958), 536–547.

Let $V_{n,k}$, $W_{n,k}$, and $X_{n,k}$ denote the Stiefel manifold of all orthonormal k-frames in an n-dimensional vector space over the real numbers, the complex numbers, and the quaternions, respectively. These manifolds may be considered as fibre bundles over spheres of dimension $n-1$, $2n-1$, and $4n-1$, respectively, having as fibre $V_{n-1,k-1}$, $W_{n-1,k-1}$, and $X_{n-1,k-1}$, respectively. This paper is concerned with the question of the existence of cross sections of these fibre bundles. The bundle $V_{n,k} \to S^{n-1}$ admits a cross section if and only if the sphere S^{n-1} admits a continuous field of tangent $(k-1)$-frames [previous work on this question is summarized in a recent note by the author; cf. Proc. Cambridge Philos. Soc. 53 (1957), 817–820; MR 21 #3853].

The author's first theorem asserts that if $V_{m,k-1}$ admits a cross section, then so does $V_{2m,k}$ $(m \geqq k \geqq 2)$. Since $V_{2,1}$ admits a cross section, it follows by induction that $V_{m,k}$ admits a cross section for $m=2^k$. For any integer $k \geqq 1$, let a_k denote the least value of m such that $V_{m,k}$ admits a cross section. An integer k is called regular if m is divisible by a_k whenever $V_{m,k}$ admits a cross section. It is known that k is regular for $1 \leqq k \leqq 9$; no irregular values of k are known. The author's second theorem may now be stated as follows: For any integer $k \geqq 1$, a_k is a power of two. If k is regular, $V_{m,k}$ admits a cross section if and only if m is a multiple of a_k. If k is irregular, $a_k = 2a_k'$ where a_k' is a power of two such that $k \leqq a_k' < 2k$, and $V_{m,k}$ admits a cross section if and only if $m \geqq a_k$ and m is a multiple of a_k'. The third theorem asserts that for any integer $k \geqq 1$ there exist positive numbers b_k and c_k such that $W_{m,k}$ or $X_{m,k}$ admits a cross section if and only if m is a multiple of b_k, or of c_k, respectively.

These theorems reduce the question of the existence of cross sections to the determination of the integers a_k, b_k, and c_k. This the author does for small values of k only, although he obtains upper bounds for b_k and c_k, and determines their prime factors.

The methods of proof depend heavily on the theory of the intrinsic join, developed by the author in a previous paper [see the preceding review].

W. S. Massey (Providence, R.I.)

7269:

Bing, R. H. An alternative proof that 3-manifolds can be triangulated. Ann. of Math. (2) 69 (1959), 37–65.

The following approximation theorem for 2-complexes is the central result of this paper. Let M be a triangulated

3-manifold with boundary, and ρ be a metric on M. Euclidean 3-space E^3 is a special case. Let $P\subset M$ be a closed set homeomorphic to a 2-complex, and let f be a positive continuous function on P. Then there exists a homeomorphism $h:P\to P'\subset M$, P' a polyhedron, such that $\rho(x, h(x))<f(x)$. This generalizes Bing's earlier theorem in which P was a topological surface [same Ann. (2) **65** (1957), 456–483; MR **19**, 300]. The proof is by methods of plane topology. Three other results are proved as applications. (1) M can be triangulated. (2) For any homeomorphism f of a solid tetrahedron X into E^3 and any $\varepsilon>0$, there exists a piecewise linear homeomorphism $h:X\to E^3$ such that $\rho(f(x), h(x))<\varepsilon$. (3) If P is a closed subset of M homeomorphic to a polyhedron, of dimension 3 or less, then P can be homeomorphically approximated by a polyhedron. Theorem (3) is new, while (1) and (2) represent comparatively elementary proofs of theorems due to E. E. Moise [ibid. **55** (1952), 215–222; **56** (1952), 96–114; MR **13**, 765; **14**, 72].

S. S. Cairns (Los Angeles, Calif.)

7270:

Koseki, Ken'iti. Poincarésche Vermutung in Topologie. Math. J. Okayama Univ. **8** (1958), 1–106.

The author claims to have proved the Poincaré conjecture on the S^3. The idea of his proof seems to be like this: Let $\mathfrak{M}$ be the given 3-manifold with a given triangulation satisfying the Poincaré condition that every 1-cycle is null homotopic. In the first step of the proof the 2-simplexes of $\mathfrak{M}$ are imbedded one after the other into Euclidean space E^3, so that at the ith stage of imbedding the image $f_i(\mathfrak{G}_i)\subset E^3$ of the 2-complex $\mathfrak{G}_i$ of $\mathfrak{M}$ hitherto chosen divides E^3 into spherical regions. In the second step the same process is continued, with the difference that a finite number of disks $Z_1, Z_2, \cdots, Z_q$ are suitably chosen so that $f_i(\mathfrak{G}_i)+Z_1+Z_2+\cdots+Z_q$ divides E^3 into spherical regions. Then these processes end finally with the disappearance of the disks; the 2-dimensional skeleton $\mathfrak{G}$ of $\mathfrak{M}$ is seen to be imbedded into E^3 and the proof is complete. {It is surprising that such a primitive idea could lead to the confirmation of the Poincaré conjecture. Unfortunately the paper is ill compiled; it is hard to find out even where the Poincaré condition was essentially applied, and the details of this long proof are totally unintelligible to the reviewer.}

H. Terasaka (Osaka)

7271:

Whitehead, J. H. C. On finite cocycles and the sphere theorem. Colloq. Math. **6** (1958), 271–281.

A 2-sphere in a closed connected 3-manifold M is "tame" if it is the image of $S\times\frac{1}{2}$ in a homeomorphism of $S\times I$ into M. If M contains a tame 2-sphere which is essential in M, then M is "reducible", a term suggesting the possibility of reducing M by "cutting through S and filling in the holes." Strengthening a result due to H. Kneser [S.-B. Deutsch. Math.-Verein. **38** (1929), 248–260], the author proves that M is reducible if and only if $\pi_1(M)$ is either cyclic infinite or a nontrivial free product. An auxiliary algebraic theorem gives a necessary and sufficient condition that a finitely presentable group be either cyclic infinite or a non-trivial free product.

S. S. Cairns (Los Angeles, Calif.)

7272:

Hirzebruch, Friedrich, und Hopf, Heinz. Felder von Flächenelementen in 4-dimensionalen Mannigfaltigkeiten. Math. Ann. **136** (1958), 156–172.

The authors give a complete solution of the vector- and plane-field problems on an oriented four manifold M. They bring necessary and sufficient conditions for the existence of such fields in terms of the Euler characteristic e, and the Poincaré bilinear form $S(x, y)$, defined on $H^2(M; Z)$ by the cup product. Let $W\subset H^2(M; Z)$ be the subset of elements w for which $S(w, x)\equiv S(x, x)$ mod 2 (x arbitrary in $H^2(M; Z)$), and let $\Omega\subset Z$ be the set of integers of the form $S(w, w)$, $w\in W$. Finally let τ be the index of the form S considered over the real numbers. It is then shown that: M admits an oriented plane-field if and only if $3\tau+2e\in\Omega$ and $3\tau-2e\in\Omega$; there exists a 2-field on M if and only if $e=0$ and $3\tau\in\Omega$. The manifold M is parallelizable if and only if $e=\tau=0$ and $x^2=0$ for all $x\in H^2(M; Z)$. It is also shown that M admits an almost complex structure if and only if $3\tau+2e\in\Omega$. In principle these are all second obstruction problems, the coefficient groups being known. By applying the present techniques of characteristic classes the authors precisely describe the possible second obstructions and thus obtain the above mentioned results.

R. Bott (Ann Arbor, Mich.)

7273:

Adachi, Masahisa. On the groups of cobordism Ω^k. Nagoya Math. J. **13** (1958), 135–156.

The author determines the cobordism groups Ω^k for $k=8, 9, 10, 11, 12$. He finds these groups to be $Z+Z$, Z_2+Z_2, Z_2, Z_2, and $Z+Z+Z$, respectively. In these dimensions he also describes the necessary and sufficient conditions on the Pontryagin and Stiefel numbers of a manifold for it to be cobordant to zero. In particular he verifies that the cobordism classes of an n-manifold is a topological invariant for $n\leqq 11$, $n\neq 8$.

R. Bott (Ann Arbor, Mich.)

7274:

Franz, Wolfgang. Über die Graphen der Abbildungen einer Mannigfaltigkeit in eine andere. Arch. Math. **10** (1959), 34–39.

Let $\mathfrak{M}$, $\mathfrak{N}$ be closed oriented triangulated manifolds of dimensions m, n and let f, g be mappings $\mathfrak{M}\to\mathfrak{N}$. By suitably deforming f and g one obtains f_0, g_0 whose coincidence points (i.e. points x of $\mathfrak{M}$ such that $f_0(x)=g_0(x)$) form a polyhedron of $m-n=d$ dimensions. It is shown that, when certain multiplicities are attached to its cells, this polyhedron becomes a d-cycle in $\mathfrak{M}$ whose homology class j is a homotopy invariant of f, g. The author obtains an explicit formula for j in terms of the homology intersection matrices of $\mathfrak{M}$ and $\mathfrak{N}$ and the homology homomorphisms induced by f and g.

P. A. Smith (New York, N.Y.)

7275:

Weier, Josef. Über Transformationen von Mannigfaltigkeiten der Dimensionsdifferenz 2. Math. Z. **69** (1958), 271–279.

Verf.'s Arbeit beruht auf der historischen Behauptung [H. Freudenthal, Compositio Math. **5** (1937), 299–314]: Die Homotopiegruppen $\pi_{d+k}(S^d)$ der Sphären S^d sind stabil für $d\geqq k+3$. Seinerseits will er diese Aussage ergänzen durch den Satz: Aus $\pi_{d+k}(S^d)=0$ für $d=k+2$ folgt dasselbe für $d>k+2$. In Wirklichkeit ist die Stabilität der $\pi_{d+k}(S^d)$ für $d\geqq k+2$ bekannt (a.a.O.). Damit werden Verf.'s Bemühungen hinfällig. Übrigens ist sogar bekannt: Aus $\pi_{d+k}(S^d)=0$ für $d=k+1$ (und bei geradem k schon für $d=k$) folgt $\pi_{d+k}(S^d)=0$ für $d\geqq k+1$ (bzw. k). Verf.'s Beweis ist eine Komplizierung des a.a.O. gegebenen. — Der Rest der Arbeit ist ein Versuch, in einem Spezialfall das zu definieren, was seit drei Jahr-

zehnten als Hopfscher Urbildzyklus bei Mannigfaltigkeitsabbildungen zum eisernen Bestand der Topologie gehört. *H. Freudenthal* (Utrecht)

7276:

Floyd, E. E.; and Richardson, R. W. An action of a finite group on an n-cell without stationary points. Bull. Amer. Math. Soc. **65** (1959), 73–76.

Let I be the group of rotational symmetries of the icosahedron, a subgroup of the group R of proper rotations of E_3. It is shown that R/I has the same homology groups over the integers as a 3-sphere and that I acts simplicially on R/I in such a way as to admit just one stationary point. A further construction leads to a combinatorial n-cell on which I acts without any stationary point whatever. *P. A. Smith* (New York, N.Y.)

7277:

Coxeter, H. S. M. Map-coloring problems. Scripta Math. **23** (1957), 11–25 (1958).

A lucid expository account of the colouring of maps on the plane, the sphere and the higher surfaces, and of its connexion with the theory of graphs. The proofs have been kept simple and readable. The paper gives a good bird's eye view of the whole subject, and ends with a useful list of references. The topics treated are as follows: The four colour conjecture; the three colour conjecture for graphs; Petersen's graph; Euler's formula; Heawood's six colour theorem for the plane; orientable and non-orientable surfaces; Tietze's six colour theorem for the projective plane; Heawood's formula for the higher surfaces and his seven colour map on the torus; Franklin's six colour theorem for the Klein bottle; the surfaces for which Heawood's formula is best possible. A proof of Heawood's five colour theorem for the plane is not given. *G. A. Dirac* (Hamburg)

7278:

Berge, Claude. Sur le couplage maximum d'un graphe. C. R. Acad. Sci. Paris **247** (1958), 258–259.

If S is a set of vertices of a graph G of order n let $p_i(S)$ be the number of components of odd order of the subgraph generated by the vertices not in S. Write

$$\xi = \operatorname*{Max}_{S\subset X}(p_i(S)-|S|),$$

where X is the set of all vertices of G and $|S|$ denotes the number of members of S. Define a coupling W of G as a set of edges such that each vertex is incident with at most one member of W, and a maximal coupling as one for which $|W|$ has the greatest possible value. The author shows that the number of edges of a maximal coupling is given by $\frac{1}{2}(n-\xi)$. Hence he deduces that the maximum number $\alpha(G)$ of independent vertices of G (no two joined) satisfies $\alpha(G)\leqq\frac{1}{2}(n+\xi)$. *W. T. Tutte* (Toronto, Ont.)

7279:

Ore, Oystein. Studies on directed graphs. III. Ann. of Math. (2) **68** (1958), 526–549.

In two preceding papers [same Ann. **63** (1956), 383–406; **64** (1956), 142–153; MR **17**, 1116; **18**, 143] the author has established conditions for the existence in a given directed graph R of a subgraph with given local degrees. In the present paper he examines the relations between such subgraphs. He proceeds by an appropriate modification of the method of alternating paths used by J. Petersen for undirected graphs. Here an alternating path with respect to a subgraph H has edges alternately in H and its complementary subgraph, and consecutive edges of the path are oppositely directed. The author shows that any two subgraphs with the same local degrees can be transformed into one another by operations on cyclic alternating paths of even length.

Starting with a "central" vertex a_0 and a given H we obtain a set of "accessible vertices", that is, vertices which can be reached from a_0 by alternating paths whose first edge is not in H. The author shows how the accessible sets and some related "cyclically equivalent sets" can be expressed in terms of the "critical sets" of his earlier papers. This leads him to a new proof of the general subgraph theorem.

In the second part of the paper the author discusses what freedom of choice exists in the selection of the edges of a subgraph which is to have given local degrees.

In the third part he points out that his theory of subgraphs can be generalized from graphs to arbitrary matrices with real non-negative elements. For example, conditions can be obtained for the existence of an additive matrix decomposition with prescribed marginal sums. *W. T. Tutte* (Toronto, Ont.)

7280:

Ore, Oystein. Conditions for subgraphs of directed graphs. J. Math. Pures Appl. (9) **37** (1958), 321–328.

The author obtains conditions for a given oriented graph to have a subgraph with specified local degrees. The local degrees $\alpha(v)$ and $\alpha^*(v)$ at a vertex v are the numbers of edges issuing from and entering v, respectively. The main theorem is as follows. Let a family of multiplicities $\kappa(v)$, $\kappa^*(v)$ be associated with the vertices of a graph R. Then R has a subgraph with these local degrees if there is a constant $c\geqq 1$ and integers d and D such that

$$d\leqq\sum_A \alpha(v)-c\sum_A \kappa(v)\leqq D,$$

$$d\leqq\sum_A \alpha^*(v)-c\sum_A \kappa^*(v)\leqq D,$$

for each set A of vertices of R, and such that $D-d<c$.

The other criteria derived in the paper are special cases of the main theorem. The theory is used to obtain a condition that R shall have a given number of regular subgraphs of the first degree without common edges. *W. T. Tutte* (Toronto, Ont.)

7281:

Harary, Frank. On the group of a graph with respect to a subgraph. J. London Math. Soc. **33** (1958), 457–461.

This paper is one of a series on enumeration problems for graphs. It gives a formula for the number of dissimilar occurrences in a graph G of k indistinguishable copies of a given subgraph H. Here "dissimilar" means "not equivalent under the group of automorphisms of G". The author uses this formula to obtain an expression for the number of non-isomorphic 2-complexes with a given 1-skeleton. *W. T. Tutte* (Toronto, Ont.)

7282:

Clarke, L. E. On Cayley's formula for counting trees. J. London Math. Soc. **33** (1958), 471–474.

A labeled tree with n points is a tree in which each point is distinguished from all the others. Cayley showed that the number of such trees is n^{n-2}. Other interesting derivations of this result have since been given by: Husimi [J. Chem. Phys. **18** (1950), 682–684; MR **12**, 467], Neville [Proc. Cambridge Philos. Soc. **49** (1953), 381–385; MR **14**, 1000], Pólya [Acta Math. **68** (1937), 145–254], Prüfer [Arch. Math. Phys. (3) **27** (1918), 142–144], and Riordan [Acta Math. **97** (1957), 211–225; MR **19**, 831],

among others. The present paper obtains the following refinement of Cayley's formula. If N_k is the number of such trees in which there are exactly k lines incident to a specified point, then

$$N_k=\binom{n-2}{k-1}(n-1)^{n-k-1};\ k=1, 2, \cdots, n-1.$$

This gives still another proof of Cayley's formula since obviously $\sum_{k=1}^{n-1} N_k=n^{n-2}$. The result also yields a well-known formula of analysis: if $z=\sum_{n=1}^{\infty}n^{n-1}w^n/n!$, where $|w|\leq 1/e$, then $w=ze^{-z}$. *F. Harary* (Princeton, N.J.)

DIFFERENTIAL GEOMETRY, MANIFOLDS

See also 7075, 7575.

7283:

Chen, Hsing-lin. On the existence theorem of the quadruply orthogonal systems of hypersurfaces in the four dimensional Euclidian space. Advancement in Math. 2 (1956), 678–689. (Chinese)

7284:

Fundo, Kr. Etude des développées et des développantes dans l'espace. Advancement in Math. 4 (1958), 249–273. (Chinese)

7285:

Voss, K. Über eine spezielle Klasse von Nabelpunkten. Comment. Math. Helv. 33 (1959), 81–88.

Der Verfasser beschäftigt sich in der vorliegenden Arbeit mit dem Index j isolierter sphärischer Punkte. Ist O ein isolierter sphärischer Punkt einer Fläche F, dann wird die Homotopieklasse welche zu der grösseren Hauptkrümmung gehört, durch dieses j charakterisiert, welches den 2πten Teil der bei einer Umgehung von O auftretenden Winkelveränderung ausmacht. Es wird gezeigt, dass im Falle wo F stetig differenzierbar, $F-O$ zweimal stetig differenzierbar ist, und es gilt für die zwei Hauptkrümmungen k_1 und k_2 von $F-O$ die Relation $(k_1-c)(k_2-c)>0$, so ist $j\leq 0$ (c eine Konstante).

Im abschliessenden Teil seiner Arbeit gibt der Verfasser solche Flächen F an, bei welchen der Index j eines isolierten sphärischen Punktes O einen beliebigen zulässigen Wert annehmen kann, welcher kleiner als 1 ist.

A. Rapcsák (Debrecen)

7286:

Picasso, E. Su particolari correlazioni definite dagli iperpiani cuspidali di una superficie non parabolica di S_4. II. Rend. Sem. Fac. Sci. Univ. Cagliari 27 (1957), 182–191.

[For part I, see same Rend. 26 (1956), 147–155; MR 19, 450]. At each non-parabolic point of a surface in a 4-dimensional space S_4 two cuspidal hyperplanes $S_3^{(i)}$ ($i=1, 2$) are defined.

A differential element E_2 determines its plane π and two characteristic lines p, q of the cuspidal hyperplanes. Relations between these lines, the (analogues of the) Lie quadrics and the polarity already introduced by the author are investigated. *E. Bompiani* (Rome)

7287:

Cherep de Guber, Rebeca. Developable surfaces related in an affine manner with a space curve. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. Serie Segunda. Rev. 5 (1957), 301–308. (Spanish. English summary)

Let $x=x(s)$ be the vector equation of a curve C referred to its arc length, $\xi(s)$ a point and $e(s)$ a unit vector of space, both referred to the fundamental affine trihedral of C. The author finds necessary and sufficient conditions that the ruled surface with equation $y(s, \lambda)=x+\xi+\lambda e$ shall be developable. If ξ and e are constant, these conditions state that the curvature k and torsion t satisfy $Ak+Bt+C=0$, with A, B, C constants. Four special cases in which ξ and e are constant are considered. In the first, for instance, e has a fixed position parallel to the osculating plane and the constants A, B, C are considered given. The equation satisfied by the coordinates of ξ is determined and then the components of e.

A. Schwartz (New York, N.Y.)

7288:

Creangă, Ioan. Sur les réseaux de Peterson dans les correspondances ponctuelles entre les variétés non holonomes. An. Şti. Univ. "Al. I Cuza" Iaşi. Secţ. I (N.S.) 3 (1957), 165–170. (Russian and Romanian summaries)

The Russian mathematician K. Peterson has shown (1866) that if a point correspondence between two surfaces does not conserve a family of asymptotic lines, then there exists, under certain regularity conditions, on one surface a conjugate net which transforms into a conjugate net on the other surface. These are called the 'fundamental' nets of Peterson.

In the present paper the author extends this idea to a point correspondence between two anholonomic manifolds V_3^2 and $\bar{V}_3^2$ situated in the euclidean spaces S_3 and $\bar{S}_3$, respectively. Since there exist two kinds of conjugate directions in a V_3^2 there exist also two kinds of such fundamental nets, whose equations are given explicitly. Finally two invariants of the given correspondence are determined. *R. Blum* (Saskatoon, Sask.)

7289:

Derchain, C. Sur les congruences de sphères de Ribaucour. Bull. Soc. Roy. Sci. Liège 27 (1958), 272–279.

Der Verfasser beschäftigt sich in dieser Arbeit mit der Bestimmung der Ribaucourschen sphärischen Kongruenzen. Eine zweiparametrige sphärische Menge des Euklidischen Raumes ist eine Ribaucoursche sphärische Kongruenz, falls die Krümmungslinien der zwei Hüllflächen sich entsprechen. Das aufgeworfene Problem ist das folgende: Zu einer gegebenen Fläche (M) soll diejenige Ribaucoursche sphärische Kongruenz bestimmt werden, für welche die gegebene Fläche Hüllfläche ist. Dazu ist die Lösung eines Differentialgleichungssystems erforderlich, welches vom Verfasser mittels einer in der Arbeit von Plumier und Rozet, [dasselbe Bull. 25 (1956), 347–356; MR 18, 923] dargelegten Methode behandelt wird. Die durch die Mittelpunkte der Sphären der sphärischen Kongruenz bestimmte Fläche (C) wird eingehend untersucht, und es werden mehrere sich daran knüpfende Fragen erörtert. *A. Rapcsák* (Debrecen)

7290:

Takasu, Tsurusaburo. Erweiterung des Erlanger Programms durch Transformationsgruppenerweiterungen. Proc. Japan Acad. 34 (1958), 471–476.

Der Verfasser stellt sich die Aufgabe, das Kleinsche Erlanger Programm zu verallgemeinern "durch Erweiterungen der klassischen Transformationsgruppen durch entsprechende Transformationsgruppen, bei denen die Gruppenparameter Funktionen von Koordinaten sind". Dieses allgemeine Verfahren soll in Zusammenhang gebracht werden mit der Formulierung der Relativitätstheorie des Verfassers, welche als eine 3-dimensionale er-

weiterte Laguerresche Geometrie aufgefasst ist [Takasu, dieselben Proc. **31** (1955), 606–609; Yokohama Math. J. **3** (1955), 1–52; MR **17**, 676; **18**, 363]. *H. Rund* (Durban)

7291:

Gheorghiu, O. Em. Objets géométriques de loi fractionnaire. Acad. R. P. Romîne. Baza Cerc. Şti. Timişoara. Stud. Cerc. Şti. **3** (1956), no. 3–4, 9–13. (Romanian. Russian and French summaries)

The author considers in X_n a geometrical object $\Omega_k(x^1, x^2, \cdots, x^n)$ $(k=1, 2, \cdots, N)$, which transforms under an allowable change of coordinates $\bar{x}^\alpha=\bar{x}^\alpha(x^1, x^2, \ldots, x^n)(\alpha=1, 2, \cdots, n)$, according to the 'fractionary' law

$$\bar{\Omega}_k(\bar{x}^\alpha)=\frac{\sum_{j=1}^N A_{j+k-1}\Omega_j(x^i)+A_{N+2-k}}{\sum_{j=1}^N A_{j+1}\Omega_j(x^i)+A_1}\quad (k=1, 2, \cdots, N),$$

where the indices are taken modulo $N+1$ and the A's are functions of the $\partial\bar{x}^\alpha/\partial x^i=p_i^\alpha$. He determines in this case the A's, which turn out to be functions of $p=\det p_i^\alpha$ and depend in addition upon $N+2$ arbitrary constants.

R. Blum (Saskatoon, Sask.)

7292:

Nožička, F. Sur le contact des hypersurfaces dans un espace affine. Rev. Math. Pures Appl. **1** (1956), no. 3, 85–90.

Contact of order k between two hypersurfaces of an affine-euclidean space is defined, and necessary and sufficient conditions for contact of order up to 4 are given. Various examples are considered.

A. G. Walker (Liverpool)

7293:

Gheorghiu, Gh. Th. Sur les variétés non holonomes de l'espace S_3. Rev. Math. Pures Appl. **2** (1957), 501–508.

$Adx+Bdy+Cdz$: forme de Pfaff à coefficients A, B, C continus et à dérivées premières $A_x, A_y, \cdots, C_z$ continues sur un ensemble ouvert V de S_3. $\mathfrak{V}$: variété de Pfaff définie par V et A, B, C. P: point générique de V. $d=d(P)$: distance de l'origine O au plan $A(X-x)+B(Y-y)+C(Z-z)=0$. $\Omega=A(B_z-C_y)+B(C_x-A_z)+C(A_y-B_x)$. $K_1=K_1(P)$: première courbure totale de $\mathfrak{V}$ en P définie, à partir du théorème de Meusnier généralisé, comme produit des courbures principales. $K_2=K_2(P)$: seconde courbure totale définie au moyen de l'indicatrice des normales et égale au produit des courbures normales des lignes de courbure. Dans le cas holonome $\Omega=0$, $K_1=K_2$ est la courbure totale en P de la surface intégrale passant par ce point. $P'=T(P)$: transformation affine de S_3 sur lui-même, donc à déterminant $\Delta\neq0$. Par utilisation des formules exprimant K_1 et K_2 en fonction de A, B, C et leurs dérivées premières, il est montré que dans une centro-affinité $(O=T(O))$ $K_id^{-4}=\Delta^2K_i'd'^{-4}$, $i=1, 2$, et que dans toute affinité $K_1/K_2=K_1'/K_2'$. Les variétés $\mathfrak{V}_a$, $\mathfrak{V}_b$ et $\mathfrak{V}_c$ sont définies comme des variétés $\mathfrak{V}$ telles que (a) $K_1d^{-4}=$Cte, (b) $K_2d^{-4}=$Cte, (c) $K_1/K_2=$Cte, respectivement. L'auteur se propose, dans un article ultérieur, d'élucider les relations entre les variétés $\mathfrak{V}_a$ et $\mathfrak{V}_b$ d'une part, et les variétés du type Ţiţeica-Wilczyński introduites par T. Mihăilescu [Acad. R. P. Romîne. Stud. Cerc. Mat. **6** (1955), 175–192; MR **17**, 527] d'autre part. L'article termine sur quelques considérations sur les variétés $\mathfrak{V}_c$ lorsque A, B, C sont des fonctions linéaires et homogènes de x, y et z.

Chr. Pauc (Nantes)

7294:

Inagaki, Masaru. On the polar projection with respect to normal curves. Math. Mag. **31** (1957/58), 141–153.

In S_n siano: P un punto generico, C^n una curva razionale normale, $P_1, P_2, \cdots, P_n$ gli n punti in cui C^n è osculata da iperpiani passanti per P. Proiettando C^n da un S_{n-m-1} $(n-m)$-segante sopra un S_m $(n\geqq m)$ si ottiene ivi una C^m razionale normale. Associando al punto P le proiezioni P_i' dei punti P_i si ottiene una corrispondenza tra i punti di S_n e le n-ple di punti di C^m. L'Autore esamina le principali proprietà di questa corrispondenza e ne indica alcune applicazioni.

D. Gallarati (Genoa)

7295:

Vogel, Walter O. Regelflächen in Riemannschen Mannigfaltigkeiten. Math. Z. **70** (1958), 193–212.

By a ruled surface of a Riemannian manifold V_n we understand a one-parameter family of geodesics of V_n, i.e., the surface formed of these geodesics which are generators of the ruled surface. The principal purpose of the paper is to study the lines of striction (Kehllinien) of the ruled surfaces of V_n, and especially to generalize some theorems concerning the lines of striction of the ruled surfaces of euclidean space R_3 to those of V_n, as well as to deduce the fundamental equations for the ruled surfaces of V_n. The author first gives the analytic expressions of the ruled surfaces of V_n and of their lines of striction, then deals with the lines of striction, of which there are three types: (1) point-formed (degenerated) lines of striction; (2) the envelope of the generators (edge of regression); (3) the general one, whose tangents do not coincide with the directions of the generators. The generalisations of the theorems of Bonnet, Beltrami, Pirondini and Darboux hold also for ruled surfaces of V_n with lines of striction of type 3, i.e.: (1) A line of striction is characterized by the fact that the tangents to the generators along the line of striction are V_2-parallel in the sense of Levi-Civita. (2) A line of striction is the locus of points at which the geodesic curvature of a family of isogonal lines (with arbitrary but constant angle $\neq0$) to the family of generators vanishes. (3) If a curve on a ruled surface of V_n is two of the following (a) geodesic line of the surface, (b) isogonal trajectory of the generators, (c) line of striction of type 3; then it is the third. (4) If one rotates the tangents of the generators at the points of a line of striction on a ruled surface of V_n through an arbitrary but fixed angle, then the rotated tangents form the starting direction of generators of a second ruled surface which touches the first ruled surface along its line of striction and has this curve as the line of striction. At the end there are two examples, viz. $V_n=R_3$ and $V_n=V_2$ in R_3.

A. Kawaguchi (Sapporo)

7296:

Satô, Isuke. On Riemannian manifolds of separated curvature. Tôhoku Math. J. (2) **10** (1958), 130–134.

An n-dimensional Riemannian space whose curvature tensor can be written in the form $R_{ijkl}=\sigma S_{ij}S_{kl}$, where σ is a non-zero scalar, is called a manifold of separated curvature [Matsumoto, Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **27** (1952), 175–188; MR **14**, 903]. In the present paper several theorems are derived which serve to characterise such spaces; for instance, a necessary and sufficient condition that the space be of separated curvature is that at each point the index of nullity is $n-2$ [for the latter concept see Chern and Kuiper, Ann. of Math. (2) **56** (1952), 422–430; MR **14**, 408]. Conditions that a space of separated curvature be one of recurrent curvature are derived.

H. Rund (Durban

7297:

Dolbeault-Lemoine, Simone. Réductibilité de variétés plongées dans un espace à courbure constante. C. R. Acad. Sci. Paris **247** (1958), 1705–1707.

This paper considers a locally reducible Riemannian V_{n-1} imbedded in a V_n of constant curvature. If $n\geqq 4$ and the curvature of V_n is positive, the metric of V_{n-1} is the sum of either (1) two metrics of constant positive curvature, or (2) a metric of constant positive curvature and the square of an exact differential. If the curvature of V_n is negative and if V_{n-1} is not locally Euclidean, the metric of V_{n-1} is the sum of either (1) two metrics of constant curvature, one positive and one negative, or (2) a metric of constant curvature (positive or negative) and the square of an exact differential.

This result is applied to a locally reducible V_{n-1} imbedded in S_n. For $n\geqq 4$ each connected component is contained in the Riemannian product of two spheres of dimension $p-1$ and $n-p$, respectively. Similarly, a V_{n-1} ($n\geqq 4$) imbedded in P_n is contained in a W_{n-1} which has a two-sheeted covering which is the Riemannian product of S_{p-1} and S_{n-p}. *C. B. Allendoerfer* (Seattle, Wash.)

7298:

Hu, Hou-Sung. On the deformation of a Riemannian metric V_m in a space of constant curvature S_{m+1}. Acta Math. Sinica **6** (1956), 320–332. (Chinese. English summary)

7299:

Hwang, Cheng-Chung. The normal coordinates o. a Riemannian manifold. Acta Math. Sinica **6** (1956), 452–463. (Chinese. English summary)

7300:

Hu, Hou-Sung. Determination of the second fundamental form of a surface V_2 in a Riemannian space V_3 by its mean curvature. Acta Math. Sinica **6** (1956), 619–630. (Chinese. English summary)

It has been shown by T. Y. Thomas [Bull. Amer. Math. Soc. **51** (1945), 390–399; MR **7**, 30] that in general the mean curvature of a surface yields an algebraic determination of its second fundamental form $b_{ij}dx^idx^j$; he has given the explicit expression of b_{ij} in terms of the metric tensor g_{ij}, the mean curvature H and their derivatives.

In this paper we consider the same problem for the surface V_2 in 3-dimensional Riemannian space V_3.

From the author's summary

7301:

Avez, André. Conditions nécessaires et suffisantes pour qu'une variété soit un espace d'Einstein. C. R. Acad. Sci. Paris **248** (1959), 1113–1115.

The author proves the following three theorems on a Riemannian manifold (whose metric may or may not be positive definite). While theorems 1 and 2 are local, theorem 3 is global.

Theorem 1 [resp. theorem 2]: A Riemannian manifold V_n is an Einstein space if and only if the mapping $\sum \varphi_\alpha\, dx^\alpha \to \sum(\nabla^\lambda\nabla_\lambda\varphi_\alpha)dx^\alpha$ is an endomorphism of the space of closed 1-forms [resp. exact 1-forms].

Theorem 3: A compact Riemannian manifold V_n is an Einstein space if and only if the mapping defined above is an endomorphism of the space of co-closed 1-forms. The Stokes theorem is used for the proof of theorem 3.

S. Kobayashi (Cambridge, Mass.)

7302:

Švarc, A. S. Homologies of spaces of closed curves. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 769–772. (Russian)

This note concerns the classically metrized space $\Omega(M)$ of oriented rectifiable loops on a Riemann manifold (M). Several other spaces are associated with $\Omega(M)$, in particular, the space of equivalence classes of loops under translation mod 1 on the parameter of the loop. The author claims conclusions of Morse and of Bott are in error as regards multiple geodesics and presents some results in this direction on the Morse type numbers. Special attention is given the m-sphere.

D. G. Bourgin (Urbana, Ill.)

7303:

Shizuma, Ryoji. Über geschlossene Geodätische auf geschlossenen Mannigfaltigkeiten. Nagoya Math. J. **13** (1958), 101–114.

The author considers the Morse theory for the space, $\tilde\Omega(M)$, of piecewise regular maps of the circle into a Riemann manifold M. From his main result he deduces that on every compact Riemann manifold there exists at least one closed geodesic. His argument is not quite clear to the reviewer, as the assertion on p. 104 that the projection $\tilde\Omega\to\tilde\Sigma$ is a fibering in the sense of Serre is false, and the covering homotopy theorem, applied to this projection, is used in the proof of theorem I. {The reviewer made a somewhat similar error in his paper, Ann. of Math. (2) **60** (1954), 248–261 [MR **16**, 276], as was pointed out by A. S. Švarc [#7302 above].} *R. Bott* (Ann Arbor, Mich.)

7304:

Knebelman, M. S. Homothetic mappings of Riemann spaces. Proc. Amer. Math. Soc. **9** (1958), 926–927.

Let M be a Riemannian manifold, H the Lie algebra of infinitesimal homothetic transformations and I the Lie algebra of infinitesimal isometries of M. The author gives a long proof of the fact that I is an ideal of H. (This is a trivial consequence of the fact that $X\to c(X)\in R$, where $X\in H$ and $c(X)$ is the dilatation constant of X, is a Lie algebra homomorphism.) Then he states that if M is a space of non-zero constant curvature, then $H=I$.

S. Kobayashi (Cambridge, Mass.)

7305:

Constantinescu, Corneliu. Mobilität und Bettische Zahlen der Riemannschen Mannigfaltigkeiten. Rev. Math. Pures Appl. **2** (1957), 435–443.

The author studies relations between the freedom of motions and the Betti numbers of a compact Riemannian manifold and he obtains the following results. In a compact Riemannian manifold with non-zero characteristic, a harmonic vector and a Killing vector are always orthogonal. In a compact homogeneous Riemannian manifold with a non-zero characteristic, the one-dimensional Betti number is equal to zero. When an n-dimensional compact Riemannian manifold with non-zero characteristic admits a group of motions such that there exists an $(n-1)$-dimensional orbit passing through an arbitrary point of the manifold, the one-dimensional Betti number of the manifold is at most equal to one. Consequently, a two-dimensional compact Riemannian manifold with a non-positive characteristic does not admit an infinitesimal motion. When an n-dimensional Riemannian manifold admits a group of motions such that there exists an $(n-1)$-dimensional orbit passing through an arbitrary point of the manifold, the one-dimensional Betti number of the manifold is at most equal to n. An n-dimensional

compact orientable homogeneous Riemannian manifold has k-dimensional Betti number at most equal to $\binom{n}{k}$. An n-dimensional compact orientable homogeneous Riemannian manifold with non-zero characteristic has k-dimensional Betti number smaller than $\binom{n}{k}$.

K. Yano (Tokyo)

7306a:

Tabata, Fujio. On the properties of a Riemannian space with a parameter $\mathfrak{S}(t)$ projected its tangent space. Bull. Kyoto Gakugei Univ. Ser. B, no. 5 (1954), 16–21.

7306b:

Tabata, Fujio. On the "relative rotation" in the field of orthogonal ennuple formed of principal directions in a "Riemannian space with a parameter" $\mathfrak{S}(t)$. Bull. Kyoto Gakugei Univ. Ser. B, no. 6 (1955), 21–29.

7306c:

Tabata, Fužio. Kinematic interpretation of Riemannian geometry with parameter. Bull. Kyoto Gakugei Univ. Ser. B, no. 7 (1955), 31–34. (Russian. Japanese summary)

7306d:

Tabata, Fujio. On the proper change of "curvatures" of a "proper" hyper-surface in a "Riemannian space with a parameter $\mathfrak{S}(t)$". Bull. Kyoto Gakugei Univ. Ser. B. no. 9 (1956), 6–11.

These four papers deal with a Riemannian space $S(t)$ with a parameter t (or time) in which the fundamental metric tensor $g_{ab}(t, u)$ {the author uses the notation $s_{ab}(t, u)$} depends on not only the point (u) but also the parameter t. In the first paper some of the local behavior of the space is investigated, by considering the behavior of the motion of orthogonal projection of a neighborhood of a point (u) upon a tangent euclidean space E at (u) for a time interval. The second paper deals with the relative rotation in the field of the orthogonal ennuples composed of principal directions in a surface $S(t)$ with a parameter t in a three-dimensional euclidean space E_3, by making use of the tensor $j_{ab}=(h_{ab}-Hg_{ab})/I$, where h_{ab} is the second fundamental tensor, H the mean curvature and I one-half of the difference of two principal curvatures. In the third paper the author tries to give a kinematic interpretation to the space $S(t)$. In the last he obtains the expressions for the (proper) change of the mean curvature and the scalar curvature, etc., of a (proper) hypersurface in $S(t)$, and discusses the necessary and sufficient conditions that the curvatures remain invariant for change of the parameter t.

A. Kawaguchi (Sapporo)

7307:

Murgescu, Viorel. Sur quelques invariants attachés à un tenseur dans les espaces à connexion affine. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. 7 (1956), no. 2, 75–98. (Romanian. Russian and French summaries)

Let A_n be a space with affine connection having the coefficients $\Gamma_{jk}{}^i$, and V a vector at the point $x^1, x^2, \cdots, x^n$ with components $v^1, v^2, \cdots, v^n$. Let the 'length' of V be defined by the analytic function $\Lambda(x^i, v^i)$ $(i=1, 2, \cdots, n)$ having the following properties: (a) It is homogeneous of degree nonzero with respect to the v^i; (b) it is invariant under parallel transport of V from x^i to x^i+dx^i; (c) its expression is invariant under coordinate transformations; (d) it is uniquely determined by the preceding conditions.

It is shown that the length thus defined exists if the Poisson brackets of the system expressing condition (b) are independent, and that it depends not only on the components of V but also on the components of the curvature tensor of A_n. In the case $n=2$ one obtains for this length an expression found previously by A. Haimovici and J. Kanitani.

Other results are: (1) Spaces A_2 which admit a Riemannian metric (i.e., for which Λ is the square root of a quadratic form in v^1, v^2) are equiaffine, i.e., they satisfy the relations $B_{ij}=R_{hij}{}^h=0$; (2) spaces A_2 which admit the notion of length of a vector, satisfying the conditions (a), (b), (c), (d), admit at least one field of parallel vectors.

The author then considers a modification of the above problem by substituting for condition (b) the condition: (b′) Under parallel transport of V, Λ changes by a factor $\rho=\rho(x^1, x^2, \cdots, x^n)$.

Finally the following problem is treated: Given in an A_2 a covariant symmetric tensor field τ_{ij}, to find a function $\mathscr{F}(x^i, \tau_{ij})$ invariant under parallel transport of the tensor from x^i to x^i+dx^i. Various forms of $\mathscr{F}$ and the corresponding fundamental invariants are given.

R. Blum (Saskatoon, Sask.)

7308:

Petrescu, Şt. Propriétés intrinsèques des espaces X_n à connexion métrique. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 9 (1957), 329–340. (Romanian. Russian and French summaries)

The author considers a Riemannian space in which a metric affine connexion with torsion is introduced. He introduces the so-called nonholonomic frame in the space and calculates the components of the torsion, of the connexion, etc. with respect to this non-holonomic frame.

He then gives some geometric interpretations of these (invariant) components and discusses the relations between these components and the Ricci coefficients of rotations.

K. Yano (Tokyo)

7309:

Su, Buchin. Certain affinely connected spaces with areal metric. Acta Math. Sinica 7 (1957), 285–294. (Chinese. English summary)

An English translation is reviewed below.

7310:

Su, Buchin. Certain affinely connected spaces with areal metrics. Sci. Sinica 6 (1957), 967–975.

The paper deals with spaces of n dimensions having two structures, (i) an "areal" metric for the measurement of a K-dimensional area, $1\leq K\leq n-1$, (ii) a set of connection parameters depending on position as well as components of K independent vectors. For $K=1$ or $n-1$ the relation between these two structures is provided by the conditions for the connection to be euclidean, in E. Cartan's terminology, which is expressed analytically by the vanishing of the covariant derivative of the 2-index metric tensor. The author points out that these conditions are contained in what he calls the "equation of connection" between the two structures. The K-fold area is defined in terms of a fundamental function $F(x, p)$ and the parameters of connection are given by $\Gamma_{jk}{}^i(x, p)$. The equations connecting them are then

$$(1) \qquad F_{.i}-F|_h{}^\alpha p_\alpha{}^j\Gamma_{ij}{}^h=0,$$

where $_{.i}\equiv\partial/\partial x^i$ and $|_h{}^\alpha\equiv\partial/\partial p_\alpha{}^h$. The author verifies that for $K=1$ or $n-1$ the equation (1), as well as

$$(2) \qquad F|_l{}^\beta\Gamma_{jh}{}^l|_i{}^\alpha p_\alpha{}^j p_\beta{}^h=0,$$

are satisfied.

He then takes as two conventions necessary for the establishment of a geometry based on a K-fold integral the two equations (1) and (2) and points out that the first and second variations of the area integral in the two cases $K=1$ and $n-1$ reduce to known expressions.

E. T. Davies (Southampton)

7311:

Su, Buchin. On the determination of certain affine connections in an areal space. Sci. Record (N.S.) **1** (1957), 195–198.

This paper is a sequel to the paper reviewed above, in which the author was concerned with the determination of affine connections in a space with an areal metric. He supposed two conventions so that the expressions for the first and second variations of the fundamental integral can be given geometrically significant forms.

In the present paper he deals with the degree of arbitrariness of such connections, proving the theorem that: The affine connections of the class in question depend upon ρ arbitrary functions of $n+mn$ arguments x^i ($i=1, \cdots, n$) and $p_\alpha{}^i$ ($\alpha=1, \cdots, m$), where $\rho=\frac{1}{2}n(n^2+n-4)+m$.

E. T. Davies (Southampton)

7312:

Bernard, Daniel. Définition globale du tenseur de structure d'une G-structure. C. R. Acad. Sci. Paris **247** (1958), 1546–1549.

Définition d'une G-structure par un espace de repères; tenseur associé à une forme tensorielle sur un fibré quelconque ou sur un espace de repères. Torsions des différentes connexions sur un espace de repères et tenseur de structure. Applications à l'intégrabilité des G-structures, au calcul des tenseurs de structures de G-structures réductibles ou équivalentes à une structure donnée.

M. F. Atiyah (Cambridge, England)

7313:

Ramacci, Maria Gabriella. Una caratterizzazione integrale delle ipersfere degli spazi euclidei. Accad. Sci. Modena. Atti Mem. (5) **15** (1957), 177–195.

Let Σ be an n-dimensional surface in euclidean $(n+1)$-space. Let γ denote any closed curve on Σ and let s be its arc length. The author proves that Σ is an n-sphere if and only if $\oint_\gamma \varphi ds=0$ for every γ. Here φ is a certain invariant of the strip on Σ determined by γ. {Let $\mathbf{T}_1, \cdots, \mathbf{T}_n$, $\mathbf{N}$ be the moving $(n+1)$-hedron of the strip. Thus $\mathbf{T}_\alpha\mathbf{N}=0$, $\mathbf{T}_\alpha\mathbf{T}_\beta=\delta_{\alpha\beta}$; $\mathbf{T}_\alpha'=-c_{\alpha-1}\mathbf{T}_{\alpha-1}+c_\alpha\mathbf{T}_{\alpha+1}-k_\alpha\mathbf{N}$; $\mathbf{N}'=\sum_1^n k_\alpha\mathbf{T}_\alpha$; $c_0=c_n=0$. If Σ is a sphere, then $k_1=$ const, $k_2=\cdots=k_n=0$. Conversely if $k_2=0$ for every γ, then $k_1=$const and Σ is a sphere. With this notation, $\varphi=k_2-(\tan^{-1}(c_1/k_1))'$. Thus, by the author's result, Σ is a sphere if $\oint_\gamma k_2 ds=0$ for every γ.}

P. Scherk (Boulder, Colo.)

7314:

Gu, Čao-Hao. Imbedding of a Finsler space in a Minkowski space. Acta Math. Sinica **6** (1956), 215–232. (Chinese. Russian summary)

7315:

Gu, Čao-Hao. The theory of affine imbedding. Acta Math. Sinica **6** (1956), 464–471. (Chinese. Russian summary)

7316:

Kuiper, Nicolaas H. Isometric and short imbeddings. Nederl. Akad. Wetensch. Proc. Ser. A. **62**=Indag. Math. **21** (1959), 11–25.

The author continues his work on the C^1 isometric imbeddings of Riemannian manifolds. His first main theorem is that if a manifold has a "uniformly essentially short" C^∞ imbedding in a Euclidean space of higher dimension, then this imbedding can be modified to become a C^1 isometric imbedding. The modification can be as small as desired, in terms of the maximum distance between new and old images of points of the manifold, and if the C^∞ imbedding is "closed", the C^1 imbedding is "closed".

His final theorem relates C^1 isometric realization to topological characteristics of 2-manifolds (surfaces): A complete orientable surface with finitely generated fundamental group has a closed C^1 isometric imbedding in E^3. If the surface is non-orientable, the imbedding is possible in E^4.

John Nash (Cambridge, Mass.)

7317:

Kreter, Reinhold. Zusammenhänge in Finslerschen Räumen. Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe **6** (1956/57), 353–365. (Russian, English and French summaries)

In this paper there is an explanation of the history of the theory of Finsler spaces which arose as generalizations of Riemannian spaces from the variation problem. Then Cartan's connection theory is sketched in fair detail. The author seems to consider Cartan's connection as the most general one defining a parallel displacement for vectors depending on direction and position. As is well known, there are many other connections defined by Berwald, Taylor, Synge, Vagner, Varga, Rund, Barthel and Laugwitz in Finsler spaces {the author seems to have no knowledge of the paper by the reviewer [Tensor (N.S.) **6** (1956), 165–199; MR **18**, 931]}. Then the invariance of the form $\Omega=g_{ik}(x, y)\xi^i\eta^k$ of any vectors ξ^i and η^i is investigated. In the course of this investigation, he discusses also the essential differences and relations between these connections. In connection with the deduction of the curvature tensors by Cartan, Rund's curvature tensor is especially treated.

A. Kawaguchi (Sapporo)

7318:

Rund, H. Some remarks concerning the theory of non-linear connections. Nederl. Akad. Wetensch. Proc. Ser. A **61**=Indag. Math. **20** (1958), 341–347.

The present paper gives some remarks concerning the theory of non-linear connections in Finsler spaces which was studied by V. Vagner [Trudy Sem. Vektor. Tenzor. Analizu **7** (1949), 65–166; MR **13**, 777] and A. Kawaguchi [Tensor (N.S.) **6** (1956), 165–199; MR **18**, 931]. Following Vagner's notations, the two kinds of connections

$$\overset{1}{\delta}X^i=dX^i+\overset{1}{\Gamma}{}^i{}_k(x, X)dx^k, \quad \overset{2}{\delta}Y_i=dY_i-\overset{2}{\Gamma}{}_{ik}(x, Y)dx^k,$$

are considered. Then — under the assumption that if X^i undergoes a parallel displacement, then so does Y_i, when $Y_i=g_{ij}(x, X)X^j$ — it is shown that the $\overset{2}{\Gamma}{}_{ik}(x, Y)$ are completely determined by the $\overset{1}{\Gamma}{}^i{}_k$ and $g_{ij}(x, X)$. The author says that the further assumption that the length of a vector remains unchanged under the parallel displacement defined by the $\overset{1}{\Gamma}{}^i{}_k$ defines the most general type of non-linear metric connection. Moreover, there are some remarks concerning the absolute differentials $\overset{1}{\delta}g_{ij}$, $\overset{2}{\delta}g_{ij}$ and geodesics.

A. Kawaguchi (Sapporo)

7319:

Moór, Arthur. Konformgeometrie der verallgemeinerten Schouten-Haantjesschen Räume. I, II. Nederl. Akad. Wetensch. Proc. Ser. A **61** = Indag. Math. **20** (1958), 94–113.

J. A. Schouten and J. Haantjes [Monatsh. Math. Phys. **43** (1936), 161–176] investigated the geometry in the spaces R_n and R_n^* in which the elements of support (u) are co- and contravariant vector densities of weights $-p$ and $+p$, respectively, and the metric tensor g_{ik} is derived from a given fundamental function $L(x, u)$. These two spaces R_n and R_n^* are dual to each other. The author generalizes these S-H-spaces in such a way that the metric tensor $g_{ik}(x, u)$ is assumed to be given a priori but not to be derived from $L(x, u)$, and defines the duality of these spaces. A generalized covariant space $\mathfrak{R}_n$ is not always dualisable with a generalized contravariant space $\mathfrak{R}_n^*$. The necessary and sufficient condition for the dualisability of $\mathfrak{R}_n$ and $\mathfrak{R}_n^*$ is stated. The theory of displacement in these spaces and the conformal geometry of these generalized metrics are discussed, where the conformal transformation of the type $g_{ik} = e^{2\sigma} g_{ik}$ is taken into consideration, σ being a function not only of x but also of u. Pseudo-conformal tensors of weight h are defined and the corresponding displacement is called the pseudo-conformal displacement. When $\sigma = \sigma(x)$, the theory becomes the absolute conformal case. In the case $A_{0ik} = l_i B_k$ and $\sigma = \sigma(u)$, an absolute conformal displacement is also defined. *A. Kawaguchi* (Sapporo)

7320:

Barthel, Woldemar. Über metrische Differentialgeometrie, begründet auf dem Begriff eines p-dimensionalen Areals. Math. Ann. **137** (1959), 42–63.

The paper deals with metric differential geometry in an areal space, where an areal space means one such that, in the n-dimensional space, the p-dimensional areal element $(1 \leqq p \leqq n-1)$ is given a priori in the form $d\mathscr{A} = f(x^{i_1}, dx^{i_1}, \cdots, dx^{i_p})$, as defined by A. Kawaguchi [cf. Monatsh. Math. Phys. **43** (1936), 289–297; Tensor (N.S.) **1** (1950/51), 14–45, 67–88, 89–103; Sugaku **3** (1951), 76–81; MR **12**, 536; **13**, 384, 385, 493]. The metric differential geometry is developed from two elements, one is the p-dimensional areal element $d\mathscr{A}$ and the other the invariant differential of the form: $DX^I = dX^I + X^K \gamma_K{}^I{}_h(x, X) dx^h$ for simple p-vectors $X^I = p! X^{[i_1} X^{i_2} \cdots X^{i_p]}$. The latter, however, must satisfy the postulate that under the parallel displacement of a simple p-vector, i.e. $DX^I = 0$, its area $\mathscr{A}(x, X)$ and simpleness remain unaltered. This postulate means, of course, a link between the area and the invariant differential. In order to make clear the relationship we need the fundamental p-tensor: $g_{IK}(x, X) = \partial_{IK}{}^2(\frac{1}{2} f^2)$, which is deduced from the areal element $f(x, X)$ by making use of the the derivatives by X^I on the Grassmann's cone GK^p defined by Iwamoto [Math. Japonicae **1** (1948), 74–91; MR **10**, 482] and studied by the present author [Arch. Math. **9** (1958), 262–274; MR **20** #3236]. Because of the general construction of vector product, these fundamental concepts can be extended to the case of $(n-p)$-vector densities, i.e. invariant differential of $(n-p)$-vector densities, etc. Then the fundamental equations of the transversal $(n-p)$-vector densities of a p-surface are derived in order to obtain the curvature quantities of the p-surface. By means of these quantities the variation of the area of a p-surface is expressed in terms of its mean curvature vector and fundamental quantities of the space. When a volume element $d\mathscr{V} = \mathscr{F}(x) dx^1 \cdots dx^n$, furthermore, is given a priori, then a dual $(n-p)$-dimensional areal element is induced. The fundamental equations of the normal p-vector of a $(n-p)$-surface permit a geometrical interpretation of the first variation of the dual area of a $(n-p)$-surface. Finally the possibility of a special choice of the coefficients of the invariant differential is considered. *A. Kawaguchi* (Sapporo)

PROBABILITY

See also 7090, 7224, 7367, 7372, 7373, 7375, 7407, 7450, 7597.

7321:

Hadwiger, H. Zur Axiomatik der innermathematischen Wahrscheinlichkeitstheorie. Mitt. Verein. Schweiz. Versich.-Math. **58** (1958), 151–165.

Unter Hinweis auf das Invarianzprinzip in der Integralgeometrie wird ein axiomatischer Aufbau der Wahrscheinlichkeitstheorie skizziert, bei dem der Ereignisraum R Wirkungsraum einer Transformationsgruppe G ist. Sei E eine nicht leere Teilmenge von R und K eine Klasse von Teilmengen aus R, welche die leere Menge und mit zwei fremden Mengen auch deren Vereinigung enthält. Überdies enthalte K mit einer Menge A auch alle Mengen gA, $g \in G$. Wenn Q eine über K definierte monotone, additive und G-invariante Mengenfunktion mit $Q(E) \neq 0$ ist, $C \subset R$ und $C \cap E \in K$, dann sei $P(C) = Q(C \cap E)/Q(E)$. Sei x_i eine Folge vom Elementen aus R, $A \in K$ und $N_k(A)$ = Anzahl der $x_i \in A$, mit $1 \leqq i \leqq k$. x_i heisst G-gleichverteilt, wenn $N_k(A) \to \infty$ für jedes A und $N_k(B)/N_k(A) \to Q(B)/Q(A)$ für alle A, $B \in K$ mit $Q(A) > 0$. Für eine G-gleichverteilte Folge ist natürlich $\lim_{k \to \infty}(N_k C \cap E)/N_k(E)) = P(C)$, die Wahrscheinlichkeit von C also Limes "relativer Häufigkeiten". Anwendungsbeispiele: Kästchenproblem und "Paradoxon" von Bertrand, Kürzungsaufgabe von Čebyšev. *L. Schmetterer* (Berkeley, Calif.)

7322:

Fréchet, Maurice. Sur la distance de deux lois de probabilité. Publ. Inst. Statist. Univ. Paris **6** (1957), 183–198.

Certain definitions of the distance between two probability laws proposed by Paul Lévy are discussed. The author suggests the following modified definition: Assuming that a distance $d(X, Y)$ between two real random variables X and Y has been defined, the distance between two probability laws $F(x)$ and $G(y)$ is identified with the distance $d(X, Y)$ between two random variables X and Y having the two-dimensional distribution function $\inf[F(x), G(y)]$. In certain important cases, this gives the same result as one of the Lévy definitions. *H. Cramér* (Stockholm)

7323:

Czerwiński, Zbigniew. On the relation of statistical inference to traditional induction and deduction. Studia Logica **7** (1958), 243–264. (Polish and Russian summaries)

The paper attempts to show parallelisms between the procedures of the Neyman-Pearson theory of statistics and classical logic and to explain the lack of parallelism in certain cases. *L. J. Savage* (Rome)

7324:

Barton, D. E.; and David, F. N. Sequential occupancy. Biometrika **46** (1959), no. 1/2, 218–223.

The authors find the probability distributions and

moments of the following random variables. (a) The number of balls it will be necessary to throw until exactly k boxes remain empty, when there are n equally likely boxes, each of unlimited capacity, and the balls are thrown independently. (b) Same as (a), except that the capacity of each box is r balls, so that once a box has r balls, the probability any other ball will fall into it becomes zero. (c) Same as (a), except that at each stage the probability that a ball fall into a box depends in a special way upon the number of balls already in the box. (d) same as (a), except that each box has fixed but unequal probabilities. (e) The number of boxes that will remain empty after a fixed number of balls is thrown, when each box has fixed but possibly unequal probabilities.

Applications to constructing models for the numbers of insects trapped in boxes are mentioned.

L. Weiss (Ithaca, N.Y.)

7325:

Rényi, A. Quelques remarques sur les probabilités des événements dépendants. J. Math. Pures Appl. (9) **37** (1958), 393–398.

Das Hauptresultat der vorliegenden Abhandlung lässt sich wie folgt formulieren: $\mathfrak{A}$ sei eine von den Elementen $A_1, \cdots, A_n$ erzeugte Boolesche Algebra. B_j $(j=1, \cdots, \gamma)$, $\gamma=2^{2^n}$, sei ein beliebigen Element dieser Algebra, P^* eine Wahrscheinlichkeitsfunktion auf $\mathfrak{A}$ mit der Eigenschaft:

$$P^*(A_k)=1 \text{ oder } P^*(A_k)=0 \quad (k=1, \cdots, n),$$

b_j $(j=1, \cdots, \gamma)$ seien beliebige reelle Zahlen. Es gelte $\sum_{j=1}^{\gamma} b_j \cdot P^*(B_j) \geqq 0$ (bzw. $=0$) für beliebige Funktionen P^* mit der oben angegebenen Eigenschaft. Dann gilt für jede Wahrscheinlichkeitsfunktion P auf $\mathfrak{A}$:

$$\sum_{j=1}^{\gamma} b_j \cdot P(B_j) \geq 0 \text{ (bzw.} =0).$$

Dieser Sachverhalt gestattet erhebliche Vereinfachungen bei Beweisen von Formeln der elementaren Wahrscheinlichkeitstheorie, in denen die Wahrscheinlichkeiten linear auftreten.

H. Kiesow (Münster)

7326:

Dugué, M. D. Sur le théorème de Lévy-Cramér. Publ. Inst. Statist. Univ. Paris **6** (1957), 213–225.

Let $\varphi, \varphi_1, \cdots, \varphi_n$ denote characteristic functions, and suppose that $\varphi=\varphi_1^{\alpha_1} \cdots \varphi_n^{\alpha_n}$, where the α are positive constants (rational or not). Then if φ is normal, the φ_i are normal, and if φ is Poisson, the φ_i are Poisson. The author points out that the first statement is an unpublished result due to Yu. V. Linnik, although the methods of proof are different.

H. Cramér (Stockholm)

7327:

Barton, D. E. The matching distributions: Poisson limiting forms and derived methods of approximation. J. Roy. Statist. Soc. Ser. B **20** (1958), 73–92.

Let each of K decks of N cards be made up of k kinds of cards, so that the ith deck contains a_{ij} cards of the jth kind. The K decks may be thought of as laid out in parallel rows and the cards in corresponding positions in the various decks compared. There are many ways of defining a match between the K cards in a given position: e.g., one may say there is a match when (a) all K cards in the same position are of the same kind, (b) the card in a given position in one deck designated as the "target deck" is matched by at least one of the $K-1$ corresponding cards in the other decks, or (c) there is at least one match among the K cards in a given position. Or (d) one may count all matches between all pairs of decks. Under any definition the computation of the probability distribution function of the number of matches is likely to be formidable for even moderately large values of N, and various approximations, predominantly by continuous curves, have been considered. In this paper Poisson limits are derived for certain general classes of matching distributions, and it is shown by several numerical examples that four terms of the Poisson-Charlier (or "Charlier type B") series frequently provide an excellent fit. The asymptotic form for the number of Latin rectangles is derived as a corollary. A matching problem of Levene [Ann. Math. Statist. **20** (1949), 91–94; MR **10**, 556] is generalized: a general Poisson limit is obtained and some exact distributions are derived.

T. N. E. Greville (Kensington, Md.)

7328:

Fortet, R. Sur la détermination du spectre de l'inverse d'une fonction aléatoire et ses applications. Publ. Inst. Statist. Univ. Paris **6** (1957), 227–240.

Soit $Z=1/X$ l'inverse d'une variable aléatoire X de densité de probabilité $f(x)$. Si $f(0)\neq 0$ l'espérance mathématique de Z n'existe pas au sens classique. Mais il est possible, dans des cas étendus, de définir une espérance mathématique en valeur principale de Cauchy:

$$E\{Z\}=\text{v.p.} \int_{-\infty}^{+\infty} x^{-1}f(x)dx=\lim_{\varepsilon\downarrow 0} \int_{|x|>\varepsilon} x^{-1}f(x)dx.$$

Le calcul est explicité dans le cas où X est une variable aléatoire de Laplace-Gauss. L'auteur utilise ensuite l'idée de la valeur principale de Cauchy pour définir la covariance $\Gamma(t, \tau)=E\{Z(t)\bar{Z}(\tau)\}$ de l'inverse $Z(t)=1/X(t)$ d'une fonction aléatoire $X(t)$ et conduit explicitement les calculs dans le cas où $X(t)$ est un fonction aléatoire de Laplace-Gauss. Les résultats sont appliquées à l'étude d'un cas particulier intéressant, à savoir la détermination, en valeur principale, de $g(t)=E\{Y(t)Z(t)\}$ où $Z(t)=(\rho \cos \Omega t+U(t))^{-1}$, ρ, Ω étant des constantes, $U(t)$ une fonction aléatoire stationnaire de Laplace-Gauss, et $Y(t)$ un fonction aléatoire de Laplace-Gauss de même loi que $U(t)$.

A. Fuchs (Strasbourg)

7329:

Schulz-Arenstorff, Richard; and Morelock, James C. The probability distribution of the product of n random variables. Amer. Math. Monthly **66** (1959), 95–99.

The exact formula is calculated, yielding first the characteristic function, and then the distribution-density, of the product $y=\prod_{k=1}^{n} x_k$ of n independent random variables $x=x_k(\xi)$, their distribution laws being the following: $\rho_k(x)=1$ for $|x-a_k|\leq\frac{1}{2}$; $\rho_k(x)=0$ for any other value of x, $\rho_k(x)$ being the distribution densities and $a_k>\frac{1}{2}$.

O. Onicescu (Bucharest)

7330:

Lévy, Paul. Symétrie et dissymétrie des produits de variables aléatoires. C. R. Acad. Sci. Paris **248** (1959), 1920–1922.

"Les produits de variables aléatoires semblent n'avoir jamais fait l'objet d'une étude systématique. Dans un travail en cours d'impression, l'auteur s'est efforcé de combler cette lacune. Il indique ici quelques formules qui sont sans doute ce qu'il y a de plus nouveau dans ce travail, et résume les méthodes qu'il a employées pour l'étude de l'arithmétique multiplicative des variables aléatoires." (Author's summary)

J. Wolfowitz (Ithaca, N.Y.)

7331:

★Hanš, Otto. Generalized random variables. Transactions of the first Prague conference on information theory, statistical decision functions, random processes held at Liblice near Prague from November 28 to 30, 1956, pp. 61–103. Publishing House of the Czechoslovak Academy of Sciences, Prague, 1957. 354 pp. Kčs 34.00.

Generalities are given about stochastic maps from one metric space to another with application to E. Mourier's strong law of large numbers [Ann. Inst. H. Poincaré **13** (1953), 161–244; MR **16**, 268].

H. P. McKean, Jr. (Cambridge, Mass.)

7332:

★Hanš, Otto. Random fixed point theorems. Transactions of the first Prague conference on information theory, statistical decision functions, random processes held at Liblice near Prague from November 28 to 30, 1956, pp. 105–125. Publishing House of the Czechoslovak Academy of Sciences, Prague, 1957. 354 pp. Kčs 34.00.

Severe conditions are given for the existence of fixpoints of stochastic maps, from one metric space to another, with a view to approximating the roots of certain functional equations.

H. P. McKean, Jr. (Cambridge, Mass.)

7333:

★Hanš, Otto. Inverse and adjoint transforms of linear bounded random transforms. Transactions of the first Prague conference on information theory, statistical decision functions, random processes held at Liblice near Prague from November 28 to 30, 1956, pp. 127–133. Publishing House of the Czechoslovak Academy of Sciences, Prague, 1957. 354 pp. Kčs 34.00.

Given a mapping f depending upon a stochastic parameter from one Banach space into another, the author links the measurability of one of the maps f, f^*, f^{-1}, $(f^*)^{-1}$ to that of the other three.

H. P. McKean, Jr. (Cambridge, Mass.)

7334:

★Nedoma, Jiří. Note on generalized random variables. Transactions of the first Prague conference on information theory, statistical decision functions, random processes held at Liblice near Prague from November 28 to 30, 1956, pp. 139–141. Publishing House of the Czechoslovak Academy of Sciences, Prague, 1957. 354 pp. Kčs 34.00.

Consider mappings f from a measurable space A into a Banach space B such that the inverse map f^{-1} carries (strong) Borel subsets of B into measurable subsets of A. When B is separable, the sum of two such mappings is a mapping of the same kind. When B is not separable this is false, as the author proves.

H. P. McKean, Jr. (Cambridge, Mass.)

7335:

Badrikian, Albert. Les éléments aléatoires généralisés à valeurs dans un espace vectoriel; définitions et premiers résultats. C. R. Acad. Sci. Paris **248** (1959), 1603-1605.

Limit theorems are given for "generalized chance elements" with values in a topological vector space.

J. Wolfowitz (Ithaca, N.Y.)

7336:

Loève, Michel. A l'intérieur du problème central. Publ. Inst. Statist. Univ. Paris **6** (1957), 313–325.

The author gives proofs of and carries further results announced in a previous paper [C. R. Acad. Sci. Paris **239** (1954), 1585–1587; MR **16**, 494]. The problem is that of the analysis of limiting distributions of sums of the form $\sum_k y(X_{nk}-a_{nk})$, where $X_{n1}, X_{n2}, \cdots$ are mutually independent random variables, uniformly asymptotically negligible when $n\to\infty$, a_{nk} is a truncated expectation of X_{nk}, and y is a continuous function, vanishing at the origin. For example, it is shown that, if $cx+O(x^2)\leq y(x)\leq c'x+O'(x^2)$, for $x\to 0$, then the distribution functions of the sequence $\{\sum_k Y_{nk}, n\geq 1\}$ form a compact family, neglecting translations, if the same is true of the distribution functions of the sequence $\{\sum_k X_{nk}, n\geq 1\}$. If y does not vanish except at the origin, and if $y(x)=dx^2+O(x^2)$ for $x\to 0$, then the distribution function of $\sum X_{nk}$ is asymptotically normal with variance γ^2 if and only if that of $\sum_k Y_{nk}$ is asymptotically concentrated at the point $d\gamma^2$. This generalizes a theorem of Raikov [Izv. Akad. Nauk SSSR, Ser. Mat. **3** (1938), 323–338] who supposed that $y(x)=x^2$.

J. L. Doob (Urbana, Ill.)

7337:

★Wiener, Norbert. Nonlinear problems in random theory. Technology Press Research Monographs. The Technology Press of The Massachusetts Institute of Technology and John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1958. ix+131 pp. $4.50.

This is the first volume in a series of monographs that will consist of research studies larger in scope than a journal article, but less ambitious than a finished book. The present volume on non-linear probability problems fits very well in this category. Indeed, the results are not always definitive or complete in every mathematical detail, but they suggest further developments and are likely to stimulate work in this important field. A very attractive feature of the book is the practical background and the intuitive justifications of results that are given together with the mathematical development.

One of the most important contributions in the book can be found in lectures 2, 3, and 4. In these the author studies non-linear functionals of the Wiener process $x(t)$; it is shown how such functionals can be represented in terms of homogeneous polynomial functionals $\int K(t)dx(t)$, $\int K(t, s)dx(t)dx(s)$, etc. Using an orthogonalization procedure the representation mentioned above becomes easier to handle, and rules are given how this can be done. In this way the author is able to compute the spectrum (and covariance function) of the non-linear functional studied. While the expressions involved are complicated they are explicit, and the author shows how they can be evaluated in two situations connected with frequency modulation:

$$\exp[i\int f(t+s)dx(t)];\ \exp[i\iint f(t+s, t+\tau)dx(s)dx(t)].$$

The mathematical result is then discussed as a possible model for brain waves; the spectrum of such waves has a form that resembles the second of the frequency modulation spectra. The orthogonal expansions are also used in lectures 10 and 11 for analysis and syntheses of nonlinear networks.

Lectures 12 and 13 on coding and decoding deal with some fundamental aspects of stationary processes with discrete time. Assuming that the process is nondeterministic the author shows how it can be represented in terms of a sequence of independent and identically distributed stochastic variables. The correspondence is 1-1 between the past of the process and the past of the sequence. This representation is closely related to the

expansion of non-linear functionals in the earlier parts of the book.

In one chapter on quantum theory and two on statistical mechanics the author indicates relations between these topics and the Wiener process. The reasoning in these three chapters is less complete than in the other ones, and it is too early to know if the ideas sketched in them will be fruitfully exploited in the future.

The volume contains many other ideas and suggestions that cannot be mentioned here, but it should be clear from the above that this is a book of great interest to anybody working on stochastic processes or their applications to the natural sciences. *U. Grenander* (Stockholm)

7338:

Cheng, Shaw-lian. Harmonizable stochastic process and linear translatable stochastic functional equations. Acta Math. Sinica 8 (1958), 281–289. (Chinese. English summary)

Let $x(t)$ be a complex-valued stochastic process; the process $x(t)$ is said to be a regular stochastic process if the variance

$$E\{|x(t)-Ex(t)|^2\}$$

exists and

$$E\{|x(t+h)-x(t)|^2\}\to 0 \text{ as } h\to 0$$

for all t $(-\infty<t<\infty)$. Without loss of generality, we may assume that the process under consideration is centered at its expectation.

The process $x(t)$ is said to be harmonizable if the covariance function of $x(t)$ can be represented by

$$E\{x(t)\overline{x(s)}\}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\lambda t-\mu s)}d^2F(\lambda,\mu),$$

where $F(\lambda, \mu)$ is a function of bounded variation on the product space $R\times R$; $F(\lambda, \mu)$ is called the spectral function of $x(t)$. In this case, it is known that the process $x(t)$ can be written as

$$x(t)=\int_{-\infty}^{\infty} e^{i\lambda t}d\xi(\lambda),$$

where $\xi(\lambda)$ is a stochastic process with the covariance function

$$E\{\xi(\lambda)\overline{\xi(\mu)}\}=F(\lambda,\mu).$$

The symbolic entity "$\ \ (\lambda)$" will be called the spectrum of $x(t)$. It is zero in an open interval (λ', λ'') if $\xi(\lambda)$=constant there, and it is zero at a point λ' if it is zero in an interval $(\lambda'-\varepsilon, \lambda'+\varepsilon)$; it has an isolated point λ' with saltus $b''-b'$ if $\xi(\lambda)=b'$ in $(\lambda'-\varepsilon, \lambda')$ and b'' in $(\lambda', \lambda'+\varepsilon)$. Let S be the set of all isolated points of $d\xi(\lambda)$; it is easily seen that S is a denumerable set. If $d\xi(\lambda)$ has only zero points and isolated points, then we say that the process $x(t)$ has pure point spectrum.

Let Λ_t be a linear translatable operator [cf. T. Kitagawa, Jap. J. Math. 22 (1952), 1–18; MR 16, 151] defined on the space H of all regular stochastic process. It is known that to each Λ_t there corresponds an integral function $G(z)$ defined in complex z-plane such that $\Lambda_t e^{zt}=G(z)e^{zt}$ in $-\infty<t<\infty$. The integral function $G(z)$ is called the generating function associated with Λ_t. Denote by Q the set of roots of the equation $G(i\lambda)=0$ in the line $-\infty<\lambda<\infty$.

In this paper we have obtained theorems 3, 4 and 5, which generalize S. Bochner's results in Proc. Nat. Acad. Sci. U.S.A. 40 (1954), 289–294 [MR 15, 807].

Theorem 3. (1°) If $x(t)$ is a harmonizable stochastic process and satisfies the non-homogeneous equation

$$\Lambda_t x(t)=y(t), \tag{1}$$

where $y(t)=\int_{-\infty}^{\infty} e^{i\lambda t}d\eta(\lambda)$ is a harmonizable stochastic process. If the isolated point λ_n' of $d\eta(\lambda)$ belongs to Q, then the value of $d\eta(\lambda)$ at λ_n' must be zero.

(2°) If $y(t)$ has pure point spectrum and its point spectrum is inclosed in Q, then $y(t)=0$ and $x(t)$ has the following representation

$$x(t)=\sum_n \xi_n e^{i\lambda'_n t}, \tag{2}$$

where ξ_n is the saltus of $d\xi(\lambda)$ at λ_n'.

Theorem 4. If $y(t)$ in the non-homogeneous equation (1) is a harmonizable stochastic process and $|G(i\lambda)|$ is positive $(-\infty<\lambda<\infty)$, then the solution $x(t)$ of equation (1) is also a harmonizable stochastic process, and the spectral function $F^*(\lambda)$ of $x(t)$ and the spectral function $F(\lambda)$ of $y(t)$ satisfies the following relation

$$F^*(\lambda)=\int_{-\infty}^{\lambda}\frac{d\eta(\alpha)}{|G(i\alpha)|^2}.$$

Theorem 5. If $y(t)$ in the non-homogeneous equation (1) is a harmonizable stochastic process and the solution $x(t)$ of equation (1) is also a harmonizable stochastic process, then

$$x(t)=x_1(t)+x_2(t),$$

where $x_1(t)$ has the representation (2), and

$$x_2(t)=\int_{R-Q}\frac{e^{i\lambda t}}{G(i\lambda)}d\eta(\lambda),$$

where $d\eta(\lambda)$ is the spectrum of $y(t)$.

Author's summary

7339:

Lévy, Paul. Fonctions aléatoires à corrélation linéaire. Illinois J. Math. 1 (1957), 217–258.

"A Laplacian (normal, Gaussian) system of random variables is characterised by the following two properties: each variable has a normal distribution function and the correlation between the different variables is linear. Laplacian stochastic processes of one or several variables, or with values in an arbitrary space, are characterised by the same two properties. We propose to study here functions or systems of variables obtained by abandoning the first property and retaining only the linear correlation." (Translated from the author's introduction.)

Since the results in this lengthy paper are not easily summarized, we shall instead describe the various types of linear correlation that the author is studying. The following definitions are made. A random variable (r.v.) Y is said to depend linearly on the system X_ν if $Y=\sum a_\nu X_\nu+V$ where V is independent of the X_ν. A stochastic process $\{\Phi(t):t\in T\}$ (T is linearly ordered) is said to have linear correlation, or to belong to the class K, if whenever $\Phi(u)$ is known for $u<t$ $(u\leqq t)$, $\Phi(s)$, for $s\geqq t$ $(s>t)$, depends linearly on the known variables. Denote by K_l the class of stochastic processes $\Phi(x)$, x being an element of an arbitrary set, which are of the form

$$\Phi(x)=f(x)+\sum\phi_\nu(x)V_\nu,$$

where f and ϕ_ν are sure functions and where V_ν is a system of mutually independent r.v.'s. Let K_l^* represent the class of processes $\Phi(x)$ each of which is a sure linear function of the increments of some additive process $Z(u)$.

Let $Z(u)=f(u)+X(u)+Y(u)+S(u)$ be the decomposition of an additive process into the sum of, respectively,

a sure function, a Laplacian process, a sum of the moving discontinuities and a sum of the fixed discontinuities. It is stated (to be proven elsewhere) that K_t^* is the closure of the class of processes of the form

$$(*)\qquad \Phi(x)=\phi(x)+\int_{-\infty}^{\infty}[F(x,u)dX(u)+G(x,u)dY(u)+H(x,u)dS(u)].$$

In the present paper Lévy defines classes C, C_0, C_i ($i=1, 2, 3$) as, respectively, the class of all processes $\Phi(x)$ representable by (*), by (*) with $\phi=0$, by (*) with only the ith integral ($i=1, 2, 3$) non-zero. Each class is studied separately and at length. In particular, characterizations are given for each of these classes in terms of necessary and sufficient conditions for given functions F, G and H to be kernels in the representation (*).

R. Pyke (New York, N.Y.)

7340:

Lévy, Paul. Fonctions linéairement markoviennes d'ordre *n*. Math. Japon **4** (1957), 113–121.

Let $\varphi=\{\varphi(s)|0<s<\infty\}$ be a random function having the property that, for $0<t<\tau$, one can write $\varphi(\tau)=U(t,\tau)+V(t,\tau)$ with $U(t,\tau)$ measurable over the field generated by the $\varphi(s)$ for $s\leqq t$ and $V(t,\tau)$ independent of that field. Let $N(t,t')$ be the dimension of the linear space spanned by the $U(t,\tau)$ for $\tau\geqq t'$ (modulo constants if necessary), set $N(t)=\sup_{t'>t}N(t,t')$, and define the true order of φ to be $n=\sup N(t)$. In theorem 1 it is shown that $U(t,\tau)$ can be written as $\sum_1^n c_h(t,\tau)U_h(t)$, where the U_h satisfy relations

$$U_k(t')=\sum a_{hk}(t'|t)U_h(t)+V_k(t,t'),\ 0<t<t',$$

for suitable constants $a_{hk}(t'|t)$, the $V_k(t,t')$ being independent of the $U_h(s)$ for $s\leqq t$. If $N(t,t')\equiv n$, then (theorem 2) $\varphi(t)=\sum f_h(t)U_h(t)+V(t)$, where the $V(t)$ form an independent family of random variables and the $U_h(t)$ are the components of a vector process with independent increments. The author then considers general sums $\varphi(t)=\sum f_h(t)U_h(t)$, the $U_h(t)$ being the components of a process with independent increments; he defines the apparent order of the sum and discusses the problem of determining the true order of φ from the representation as a sum. The paper is related to two earlier papers of the author [see the preceding review and Proc. Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. II, pp. 133–175, Univ. of Calif. Press, Berkeley, 1956; MR **19**, 893], and the author points out an error in the second.

G. A. Hunt (Ithaca, N.Y.)

7341:

Cogburn, Robert. Termes variationnels des chaînes de Markov. C. R. Acad. Sci. Paris **247** (1958), 2281–2283.

Soit $X_1, X_2, \cdots, X_n, \cdots$ une chaîne de Markov homogène, c. à d. dont les probabilités de passage

$$P\{X_{m+n}\in S|X_m=x\}=P^n(x,S)$$

sont indépendantes de m. On suppose que la chaîne est à convergence exponentielle, c. à d. qu'il existe des constantes a, $b>0$ et une mesure de probabilité $\bar{P}$ telles que $|P^n(x,S)-\bar{P}(S)|\leqq ae^{-bn}$, quels que soient n, x, S.

On pose $X_{nk}=f_n(X_k)$, $k=1, 2, \cdots, k_n\to\infty$, où les f_n sont des fonctions de Borel finies. Soient X_{nk}^* les termes variationnels correspondants. L'auteur établit deux théorèmes asymptotiques relatifs à la loi de probabilité de ces termes variationnels.

A. Fuchs (Strasbourg)

7342:

Koźniewska, I. Ergodicity of non-homogeneous Markov chains with two states. Colloq. Math. **5** (1958), 208–215.

The author studies the relation between several definitions of ergodicity of a non-stationary two-state Markov chain. Let $\{p_{ij}^{(n)}\}$ be the nth transition matrix, $\lambda_n=p_{11}^{(n)}-p_{21}^{(n)}$, and $h_{ab}(m,n)$ denote the probability of a transition from state a at time m to state b at time n. A representative result is as follows. Theorem 1: Necessary and sufficient for $\lim_{n\to\infty}h_{ab}(m,n)=h_b$, $m=0, 1, 2, \cdots$, are $\sum_{i=1}^{\infty}(1-|\lambda_i|)=\infty$ and the existence of $\lim_{n\to\infty}(p_{21}^{(n)}+\sum_{i=2}^{n}p_{21}^{(i-1)}\prod_{k=i}^{n}\lambda_k)$.

J. Wolfowitz (Ithaca, N.Y.)

7343:

Braumann, Pedro. Study of a particular Markow chain. Univ. Lisboa. Revista Fac. Ci. A (2) **6** (1957/58), 281–304.

Combinatorics for a special kind of Markov chain with applications to a simple cascade process.

H. P. McKean, Jr. (Cambridge, Mass.)

7344:

Volkonskiĭ, V. A. Random substitution of time in strong Markov processes. Teor. Veroyatnost. i Primenen. **3** (1958), 332–350. (Russian. English summary)

Let $x(t,\omega)$ be a temporally homogeneous strong Markov process with right continuous paths taking values in a metric space. A process $y(t,\omega)$ is a process obtained from $x(t,\omega)$ by means of a random substitution of time if $y(t,\omega)=x(\tau_t(\omega),\omega)$, where $\tau_t(\omega)$ is monotone, right continuous in t and $\{\omega|\tau_t(\omega)\leqq s\}$ is for each t in the field generated by $x(r,\omega)$ for $r\leqq s$. Sufficient conditions are given for $y(t,\omega)$ to be a Markov or a strong Markov process. A Feller process is a process whose infinitesimal operator leaves invariant the class of continuous bounded functions. It was shown by Dynkin [Teor. Veroyatnost i Primenen. **1** (1956), 38–60; MR **19**, 691] that the infinitesimal operator A of a Feller strong Markov process continuous on the right is a contraction of a certain operator $\mathfrak{A}$ called the extended operator. Let τ be determined by $\int_0^{\tau} ds/d\varphi(x_s)=t$ where $\varphi(x)>0$ and continuous. Then if $x(t,\omega)$ and $y(t,\omega)=x(\tau_t(\omega),\omega)$ are Feller processes continuous on the right and if $\mathfrak{A}$ is the extended operator for $x(t,\omega)$, $\varphi\mathfrak{A}$ is the extended operator for $y(t,\omega)$. Assume next that the state space is a closed interval $[r_1, r_2]$. Then $x(t,\omega)$ is regular if for any x, y in (r_1, r_2) there is positive probability of hitting y when the process is started at x. It is shown that every strong Markov regular process with state space $[r_1, r_2]$ and continuous paths can be obtained from a Wiener process by means of a random substitution of time and a monotone transformation of the segment $[r_1, r_2]$.

J. L. Snell (Hanover, N.H.)

7345:

Leipnik, Roy. Integral equations, biorthonormal expansions, and noise. J. Soc. Indust. Appl. Math. **7** (1959), 6–30.

After a brief review of orthogonal and biorthogonal expansions and the Hilbert-Schmidt theory of integral equations the author shows how these concepts are relevant when studying the probability distribution of integrated functionals of Markov processes. Following Kac he uses orthogonal expansions to evaluate multiple integrals

$$\int\cdots\int K_0(x_1,x_n)K_1(x_1,x_2)\cdots K_{n-1}(x_{n-1},x_n)\,\sigma(dx_1)\cdots\sigma(dx_n),$$

especially the cases when the K's are of the form $K(x, y) = \exp(-a(x^2+y^2)+bxy)$ or $K(x,y)=\exp(-a(x+y))I_\alpha(b(xy)^{\frac{1}{2}})$ in the (x, y)-plane or first quadrant, respectively. Biorthogonal expansions are used to study the effect of instantaneous non-linear transformations upon stochastic processes. When the cross-correlation function after distortion is proportional to the cross-correlation function before distortion the process is said to have the Bussgang property. Necessary and sufficient conditions for this are discussed and exemplified. *U. Grenander* (Stockholm)

7346:

Martynov, A. V. On local infinite divisibility of Markoff processes. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 752–755. (Russian)

"As is well known, Markov processes which are homogeneous in space and time are infinitely divisible; i.e., the logarithm of the characteristic function of the transition probability of such a process is represented by a Levi-Hinčin formula. The present article considers a modification of this fact in its application to stochastic processes of general type, in particular to Markov processes which are inhomogeneous in space and time."

Author's summary

7347:

Urbanik, K. Remarks on invariant functions in Markov processes. Colloq. Math. **5** (1958), 223–230.

The author considers a continuous parameter (separable) Markov process with an at most denumerable state space X. Let $Q(x)$ denote the limiting stationary probabilities of the process. A (set-theoretic) definition is given of invariant sets and functions under shift transformations. The purpose of this paper is to study the cardinality of the range space of a given invariant function. Two theorems are proven. Theorem I: If $\sum_{x\in X} Q(x)=1$, then the range space is at most denumerable. Theorem II: For any set of non-negative numbers $q(x)$, $x \in X_0$, an at most denumerable set, satisfying $\sum_{x\in X_0} q(x)<1$, there exists a continuous parameter Markov process with an at most denumerable state space $X\supset X_0$ for which $Q(x)=q(x)$ if $x \in X_0$ and $Q(x)=0$ if $x \in X-X_0$. Moreover, there exists an invariant function of this process which has a non-denumerable range space. To prove this latter theorem a process is constructed whose state space is essentially all finite zero-one sequences, one invariant function of which has the Cantor set for its range space. {Equation (4) is incorrect. It is true whenever $E \in B_{\Omega(x)}(\tau+t)$, a condition which is satisfied where (4) is applied.} *R. Pyke* (New York, N.Y.)

7348:

Dynkin, E. B. Inhomogeneous strong Markov processes. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 261–263. (Russian)

The notion of strong Markov property (SMP) is extended to Markov processes with nonstationary transition probabilities $P(s, x; t, \Gamma)=P_{s,x}\{x_t \in \Gamma\}$. Not all the rather standard notations will be explained here but $\mathfrak{M}_{s,t}$ is the σ-algebra (Borel field) generated by x_u, $s\leq u\leq t$. The random time τ is called s-optional (independent of the future and of the s-past) if and only if (1) $\tau(w)\geq s$ for all w and (2) $\{w:\tau(w)\leq t\} \in \mathfrak{M}_{s,t}$ for every $t\geq s$. Then $\mathfrak{M}_{s,\tau}$ is the σ-algebra of sets A such that $A\cap\{w:\tau(w)\leq t\} \in \mathfrak{M}_{s,t}$ for every $t\geq s$. Two definitions of the SMP are given of which the more exigent one requires: (a) $P(s, x; t, \Gamma)$ be measurable in (s, x, t); (b) for any s, x, Γ, any s-optional τ, and any random time η which is measurable $\mathfrak{M}_{s,\tau}$ and not less than τ, we have for almost all w with respect to the $P_{s,x}$-measure:

$$P_{s,x}\{x_\eta \in \Gamma|\mathfrak{M}_{s,\tau}\}=P(\tau, x_\tau; \eta, \Gamma).$$

Simple consequences of the definition in terms of integrals are given. Now suppose that the state space is metric and the σ-algebra on it is generated by the open sets; and suppose that the Markov process has right continuous sample functions. Then the two definitions coincide; in fact, if (a) is satisfied, then (b) will hold if it holds for all η of the form $\tau+h$, where h is a positive constant. Two continuity properties of the transition functions are stated as sufficient for the SMP. No proofs.

If the state space is denumerable and the process measurable, Yuškevič [Teor. Veroyatnost. i Primenen. **2** (1957), 187–213] proved that the SMP in the form defined above holds wherever $x_\tau\neq\infty$; this result follows also from lemma 3 in a paper by the reviewer [see the following review]. *K. L. Chung* (Syracuse, N.Y.)

7349:

Chung, K. L. On a basic property of Markov chains. Ann. of Math. (2) **68** (1958), 126–149.

Let $\{x(t), 0\leq t<\infty\}$ be a Markov process, on the measure space Ω, with stationary transition probabilities and state space the set of positive integers together with $+\infty$. It is supposed that, for each t, $x(t)$ is finite with probability 1. If $(p_{ij}(\cdot))$ is the transition matrix function, it is assumed that $\lim_{t\to 0} p_{ii}(t)=1$ for all i. It is shown that, without restriction on the transition matrix function, almost all sample functions of the process may be supposed to be right lower semicontinuous. Let α be a non-negative random variable, not necessarily defined almost everywhere on Ω, which, in a sense made precise in the paper, has the property that $\alpha(\omega)\leq t$ defines a subset of Ω depending on the past of the process to time t. If $x(t)$ has the value $x(t, \omega)$ at $\omega \in \Omega$, define $y(t, \omega)=x[\alpha(\omega)+t, \omega]$, so that $y(t)$ is a function on Ω. It is then proved, along with various subsidiary results, that, relative to the domain of definition of α, $y(t)$ is measurable for $t>0$ and that the $y(t)$ process is a Markov chain, also with transition matrix function $(p_{ij}(\cdot))$. Moreover the Markov property holds in the further sense that past and future of the $x(t)$ process (the present time being α) are independent, if conditioned by finite $y(0)$. For each $t>0$, $y(t)$ is finite with probability 1. This is one of a series of papers on the effect on a Markov process with stationary transition probabilities of displacing the origin of the time parameter by a random amount. Special cases of the author's theorem have been proved by the reviewer [Trans. Amer. Math. Soc. **52** (1942), 37–64; MR **4**, 17], by the author himself [Proc. Third Berkeley Symposium Statist. Prob. vol. 2, U. of Calif. Press, Berkeley, 1956, pp. 29–40; Trans. Amer. Math. Soc. **81** (1956), 195–210; MR **18**, 941; **17**, 755] and by Yuškevič [Teor. Veroyatnost. i Primenen. **2** (1957), 187–213]. Corresponding results for uncountable state spaces have been obtained by many authors (see for example the references to this case in the last-mentioned paper), but what distinguishes the present paper from these is the fact that no hypotheses involving right-continuity are imposed on the sample functions of his processes. More recently Austin [see the following review] has obtained an important part of Chung's basic result by a simple approximation procedure, at the price of losing Chung's detailed analysis of the transition probabilities from time α to later times, which yields the absolute probability distribution of the $y(t)$ process.

J. L. Doob (Urbana, Ill.)

7350:

Austin, D. G. A new proof of the strong Markov theorem of Chung. Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 575–578.

A simple proof of a theorem of Chung [see the preceding review]. *J. L. Doob* (Urbana, Ill.)

7351:

Belyaev, Yu. K. On the unboundedness of the sample functions of Gaussian processes. Teor. Veroyatnost. i Primenen. 3 (1958), 351–354. (Russian. English summary)

It is proved that if $x(t)$ is a stationary separable Gaussian random process whose spectral function has a non-null continuous component, then almost all sample functions of the process are unbounded. An example of a process with discrete spectrum having almost all sample functions unbounded and an example of a process with discrete spectrum having almost all sample functions bounded are given. *J. L. Snell* (Hanover, N.H.)

7352:

Karlin, Samuel; and McGregor, James. Random walks. Illinois J. Math. 3 (1959), 66–81.

Consider a random walk on the non-negative integers with 1-step transition probabilities

$$(1)\qquad p_{ij}=0\ (|i-j|>1),\ p_{ij}>0\ (|i-j|=1),\ \textstyle\sum_j p_{ij}\leqq 1,$$

where, in the third part, the defect is interpreted as the probability of annihilation; let

$$(2)\qquad \pi_n=\frac{p_{01}p_{12}\cdots p_{n-1,n}}{p_{10}p_{21}\cdots p_{n,n-1}},\ n\geqq 1;\ \pi_0=1;$$

and introduce the solution Q of

$$(3)\qquad \sum p_{ij}Q_j(\xi)=Q_i(\xi)\ \ (i\geqq 0;\ Q_0=1).$$

Q_n is a real polynomial of degree n. The authors note that the operator $P:f\to\sum p_{ij}f(j)$ is symmetric with bound $\leqq 1$ on the Hilbert space of functions $f(j)$ $(j\geqq 0)$ such that $\sum f(j)^2\pi_j<+\infty$, and, in a few strokes, derive the eigen-differential expansion

$$(4)\qquad p_{ij}{}^n=\pi_j\int_{-1}^{+1}\xi^n Q_i(\xi)Q_j(\xi)\Psi(d\xi),$$

where $p_{ij}{}^n$ is the n-step transition probability from i to j and $\Psi(d\xi)$ $(\geqq 0)$ is a certain spectral measure; for walks on the unsigned integers, $\Psi(d\xi)$ is a measure from $[-1, +1]$ to 2×2 symmetric non-negative definite matrices, $Q_i(\xi)$ is a vector in 2-space, and in place of $Q_iQ_j\Psi(d\xi)$ there is the dot product $Q_i\cdot\Psi(d\xi)Q_j$. (4) is a powerful tool: for instance, it implies that $\lim_{n\uparrow+\infty} p_{ij}{}^n/p_{kl}{}^n=\pi_j/\pi_l$ if the walk is recurrent (i.e., if $\int_{-1}^{+1}(1-\xi)^{-1}\Psi(d\xi)=\sum_{i\geq 0}(p_{i,i+1}\pi_i)^{-1}=\infty$) and if $p_{jj}>0$ for some j. For announcements and related work, see Karlin and McGregor [Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 387–91; Trans. Amer. Math. Soc. **80** (1957), 366–400; MR **17**, 166; **19**, 989]. *H. P. McKean Jr.* (Cambridge, Mass.)

7353:

Pollaczek, Félix. Détermination de différentes fonctions de répartition relatives à un groupe de lignes téléphoniques sans dispositif d'attente. C. R. Acad. Sci. Paris **247** (1958), 1826–1829.

A telephone traffic model has s trunks, with arbitrary distributions of both holding-times and inter-arrival times, these times all being mutually independent. Calls finding all s trunks busy are sent away or "lost". Generating functions are found for (i) the Laplace-Stieltjes transform of the probability that the nth and the $(n+r+1)$th calls are lost, the r intervening calls are served, and these two lost calls occur within a time t of each other, and for (ii) the L.-S. transform of the probability that the nth and the $(n+r+1)$th calls are served, the r intervening calls are lost, and these two served calls occur within a time t of each other. *V. E. Beneš* (Murray Hill, N.J.)

7354:

Pollaczek, Félix. Fonctions de répartition relatives à un groupe de lignes téléphoniques sans dispositif d'attente. C. R. Acad. Sci. Paris **248** (1959), 353–355.

This is a continuation of an earlier paper [see the preceding review] on various distribution functions arising in a many-server traffic system with loss (service demands appearing when all servers are busy are dismissed, without effect on future demands, and constitute the loss). The arrivals form a renewal process with general inter-arrival distribution function, service times are independent and have the same general distribution function, and the starting conditions are arbitrary. The first function is for the state defined by: the nth arrival is lost and is separated from the next lost arrival by an interval in which n_1 demands are served, and this arrival in turn from its successor by an interval in which n_2 demands are served. In the author's usual manner, the determination of this is reduced to solution of a system of linear integral equations. In case all servers are initially idle and service times have an exponential distribution, the generating function of its Laplace transform is given explicitly, and it is noted that in equilibrium conditions the intervals between successive lost demands are mutually independent. Exponential service time distribution is essential to this latter property. Next, a distribution function is determined for the interval between the loss of the nth arrival and the next lost arrival, this interval being such that, of the r arrivals served in it, r_i on arrival find i servers busy. An explicit result is given for exponential service time distribution and equilibrium conditions. Finally a distribution function is determined for an interval in which r successive arrivals are lost, with given service state at the arrival preceding the first one lost and at the arrival succeeding the last one lost. *J. Riordan* (New York, N.Y.)

7355:

Reich, Edgar. On the integrodifferential equation of Takács. II. Ann. Math. Statist. **30** (1959), 143–148.

The author continues his work [same Ann. **29** (1958), 563–570; MR **20** #354] on the queueing model with one server, service in order of arrival, Poisson arrivals at a general rate $\lambda(t)=d\Lambda(t)/dt$, independent service-times with probability density $h(x)$, and mean service-time μ_1. The asymptotic behavior of $F(t)=\Pr\{\eta(t)=0\}$ is investigated, where $\eta(t)$=waiting-time of someone arriving at t; clearly $F(t)$=chance of not having to wait=chance that system is empty. The principal result is that if the transform of $h(x)$ is regular at infinity and around the imaginary axis, and $\limsup \mu_1[\Lambda(t)-\Lambda(u)]/(t-u)$ is less than one as $t-u\to\infty$, then $T^{-1}\int_0^T F(t)dt$ is $1-\mu_1\Lambda(T)/T+o(1)$ as $T\to\infty$. *V. E. Beneš* (Murray Hill, N.J.)

7356:

Takács, Lajos. On secondary stochastic processes generated by a multidimensional Poisson process. Magyar Tud. Akad. Mat. Kutató Int. Közl. 2 (1957), 71–80. (Hungarian and Russian summaries).

Let $\mu(S)$ be Lebesgue measure defined on the Borel

sets of a finite dimensional Euclidean space. The author considers processes $\xi(S)$ with the following properties: If $\mu(S)<\infty$: (1) $\xi(S)$ assumes nonnegative integer values and $P\{\xi(S)=0\}\neq 1$ if $\mu(S)>0$; (2) the probability distribution of $\xi(S)$ depends only on $\mu(S)$; (3) if $S_1\cap S_2=\emptyset$, then $\xi(S_1)$, $\xi(S_2)$ are independent and $\xi(S_1+S_2)=\xi(S_1)+\xi(S_2)$; (4) $\lim_{\mu(S)\to 0} P\{\xi(S)\geqq 1\}/P\{\xi(S)=1\}=1$. Assumptions 1-4 are shown to imply that

$$P\{\xi(S)=k\}=e^{-\rho\mu(S)}[\rho\mu(S)]^k/k!.$$

Secondary stochastic processes generated by such homogeneous Poisson processes are discussed in some detail. The work generalizes results obtained by the author in an earlier paper [Acta Math. Sci. Hungar. **5** (1954), 203–236; MR **16**, 938]. *M. Rosenblatt* (Bloomington, Ind.)

7357:

Driml, Miloslav; et Hanš, Otto. Trois théorèmes concernant l'expérience dans le cas continu. C. R. Acad. Sci. Paris **248** (1959), 629–631.

Conditions for the convergence to a specific value of the output of a special non-linear device; the latter depends upon a gratuitous stochastic parameter.

H. P. McKean, Jr. (Cambridge, Mass.)

7358:

Gani, J. Elementary methods for an occupancy problem of storage. Math. Ann. **136** (1958), 454–465.

Denote by the random variable $\eta(t)$ the content of an infinite dam at time t. It is supposed that $\eta(0)=u$ (positive integer), $\eta(t)$ has a jump with magnitude χ_n at time τ_n ($n=1, 2, \cdots$) and otherwise $\eta(t)$ approaches 0 with slope -1. Denote by T the instant of the first emptiness of the dam. The author determines by an elementary method the distribution, the generating function, the expectation and the variance of T in two special cases: 1) $\tau_n=n$ and $\{\chi_n\}$ is a sequence of identically distributed independent random variables with distribution

$$P\{\chi_n=k\}=p^kq \quad (k=0, 1, \cdots, p+q=1).$$

Then

$$P\{T=u+r\}=\frac{u(u+2r-1)!}{r!(u+r)!}p^rq^{u+r}.$$

2) $\{\tau_n\}$ is a Poisson process with density λ and $\chi_n\equiv 1$. Then

$$P\{T=u+r\}=e^{-\lambda(u+r)}\frac{\lambda u[\lambda(u+r)]^{r-1}}{r!} \quad (r=0, 1, \cdots).$$

L. Takács (London)

STATISTICS

See also 7585.

7359:

Hoel, Paul G. Efficiency problems in polynomial estimation. Ann. Math. Statist. **29** (1958), 1134–1145.

Given the regression model

$$E(y_i)=\beta_0+\beta_1x_i+\cdots+\beta_kx_i^k, \quad i=1, 2, \cdots, n,$$

the minimum variance unbiased estimates for the β's are given by

$$\beta=(X'S^{-1}X)^{-1}X'S^{-1}y,$$

where X is the matrix of powers of the x_i, y the matrix of the y_i and S is the covariance matrix of the y's. The generalized variance of these estimates is

$$GV(\beta)=|X'S^{-1}X|^{-1}.$$

For classical regression (S is a diagonal matrix with elements σ^2), the GV of the estimates of the ordinates of a polynomial regression curve will be minimized when $GV(\beta)$ is a minimum. The range of x's should be as great as possible; the values of the x's at which observations are taken are given by the zeros of a tabulated polynomial; equal numbers of observations should be taken at each of these points.

Comparisons of the asymptotic GV (large n) of doubling the points within a selected range, doubling the range or simply replicating the experiment were then made for: (i) uncorrelated variates; (ii) variates following a particular stationary stochastic process; (iii) variates following a pure birth stochastic process (non-stationary). Numerical results were then computed for $n=10$, showing that the asymptotic ratios of GV's are poor approximations for (ii) and (iii). The major conclusion was that, if the range cannot be extended, it is more efficient to replicate the experiment than to double the number of points within a selected range, especially if the variables are strongly correlated. *R. L. Anderson* (Raleigh, N.C.)

7360:

DeGroot, Morris H. Unbiased sequential estimation for binomial populations. Ann. Math. Statist. **30** (1959), 80–101.

"Criteria are developed for the selection of an appropriate sampling plan for the family of binomial distributions. The value of a given function, $g(p)$, is to be estimated. The problem considered here is that of determining a sampling plan and an unbiased estimator of $g(p)$ that are optimal, in some sense, at a specified value, p_0, of p. Optimality will depend, not only on the variance of the estimator, but also on the average sample size of the plan. A sampling plan, S, and estimator, f, will be considered optimal at p_0 if, among all procedures with average sample size at p_0 no larger than that of S, there does not exist an unbiased estimator with smaller variance at p_0 than that of f. In section 3 it is shown that the single sample plans and the inverse binomial sampling plans are the only ones that admit an estimator that is efficient at all values of p. In section 8 a new characterization of completeness is given for bounded sampling plans and it is shown that the dimension of the linear space of unbiased estimators of 0 can be determined simply by counting the number of boundary points. It is further shown that for a wide class of plans, the estimators that are efficient at a given value of p do not have uniformly minimum variance, although non-trivial uniformly minimum variance estimators do exist." (From the author's summary)

J. Wolfowitz (Ithaca, N.Y.)

7361:

Moshman, Jack. A method for selecting the size of the initial sample in Stein's two sample procedure. Ann. Math. Statist. **29** (1958), 1271–1275.

Let N_0^* be a first sample size which minimizes the expected total sample size in Stein's two-sample procedure. The method proposed is to choose the first sample size, N_0, to be an integer which maximizes a weighted average of the following two differences: (a) between the expected total sample sizes corresponding to first sample sizes N_0 and N_0^*; (b) between the pth percentiles of the total sample size distributions corresponding to these first

sample sizes. p is assumed to be an upper percentile given by non-statistical considerations. The weighted average depends on the unknown variance of the observations. A roughly defined "good" property of the method is implied, but precisely what optimum property the method may possess is not made clear.

M. Skibinsky (Upton, N.Y.)

7362:

Kupperman, Morton. Probabilities of hypotheses and information-statistics in sampling from exponential-class populations. Ann. Math. Statist. **29** (1958), 571–575.

Let E be an event which can occur if and only if one of the set $H_1, \cdots, H_r$ of exhaustive and incompatible populations of the discrete exponential class occurs, and let α_i be the a priori probability of H_i. The author proves that $P(H_m|E)\geqq P(H_n|E)$ if and only if $\hat{I}_m\leqq\hat{I}_n+\log(\alpha_m/\alpha_n)$, with both relations being equalities or strict inequalities together. Here $\hat{I}$ is the so-called Kullback-Leibler information-statistic for discriminating between H_j and the maximum likelihood population (based on the observed sample) if the latter is true. Applications and extensions to the case of continuous exponential class populations are discussed.

R. A. Leibler (Princeton, N. J.)

7363:

Roy, J. Step-down procedure in multivariate analysis. Ann. Math. Statist. **29** (1958), 1177–1187.

This paper presents a step-by-step procedure for determining statistical test criteria (1) for (model I) multivariate analysis of variance, and (2) for the comparison of covariance matrices, in samples from a normal multivariate distribution. The procedure calls for an a priori ordering of the variables in the population in "order of importance". The compound hypothesis regarding the normal multivariate distribution is decomposed into a sequence of sub-hypotheses. The first in the sequence concerns the marginal distribution of the first variable. The second concerns the conditional distribution of the second variable for a given value of the first. The third pertains to the conditional distribution of the third variable given the first two, and so on. Standard univariate statistical tests exist for testing each of the sub-hypotheses. The compound hypothesis is accepted if and only if all of the sub-hypotheses are accepted. The tests for the sub-hypotheses are independent if the compound hypothesis is true. A similar step-by-step procedure was devised in 1957 by S. N. Roy and R. E. Bargmann for testing the hypothesis of mutual independence of sets of variables in a normal multivariate distribution. Unlike the usual maximum likelihood method for determining overall test criteria for these compound hypotheses, these step-by-step methods do not yield overall tests which are invariant under a permutation of the variables in the population.

S. S. Wilks (Princeton, N.J.)

7364:

Roy, S. N.; and Potthoff, R. F. Confidence bounds on vector analogues of the "ratio of means" and the "ratio of variances" for two correlated normal variates and some associated tests. Ann. Math. Statist. **29** (1958), 829–841.

Consider a $2p$ variate normal population, where the ith and $(p+i)$th variates are comparable, say the same characteristic before and after the application of some treatment to the individuals in the population. Confidence bounds are derived for the appropriate ratios of parameters in the following cases. (i) The ratio of variances σ_1/σ_2 when $p=1$. (ii) Let Σ_{11}, Σ_{22} denote the covariance matrices of the first and second sets of p variates, respectively, Σ_{12} the covariance matrix of the first set with the second. Let μ, ν be non-singular matrices such that $\Sigma_{11}=\mu\mu'$, $\Sigma_{22}=\nu\nu'$, $\Sigma_{12}=\mu\rho\nu'$, where ρ is the diagonal matrix of canonical correlations between the first and second sets of variates. Simultaneous confidence bounds are found for the largest and smallest characteristic roots of the matrix $\mu\nu^{-1}\nu'^{-1}\mu'$. (iii) The ratio of means ξ_1/ξ_2 when $p=1$. (iv) Simultaneous confidence bounds for the ratios ξ_i/ξ_{p+i} ($i=1$ to p).

S. W. Nash (Vancouver, B.C.)

7365:

Rao, C. Radhakrishna. Some problems involving linear hypotheses in multivariate analysis. Biometrika **46** (1959), no. 1/2, 49–58.

Consider p correlated normal variables with the specification: (I) $E(\mathbf{y})=\mathbf{A}\boldsymbol{\tau}$, where $\mathbf{y}$ is a $p\times 1$ matrix of the random variates, $\boldsymbol{\tau}$ is an $m\times 1$ matrix of unknown parameters, and A is a $p\times m$ matrix of known coefficients; (II) the dispersion matrix of the y's is $e^{-1}\boldsymbol{\Lambda}$, where e is a known constant and $\boldsymbol{\Lambda}$ is an unknown non-singular matrix for which an estimate $\mathbf{S}$ is available with a Wishart's distribution of f degrees of freedom independently of the y's. On the basis of a given realization of the y's and $\mathbf{S}$ the author examines the following problems. (a) Is the specification (I) adequate? (b) If (I) is true, how can estimates of the τ's be obtained and their precision expressed? (c) Assuming (I), how can general linear hypotheses concerning the τ's be obtained? (d) How can simultaneous confidence limits to a class of linear functions of the τ's be obtained? Problems (a), (b), and (c) are formulated in a manner equivalent to others already considered [C. R. Rao, *Advanced statistical methods in biometric research*, Wiley, New York, 1952; MR **14**, 388; pp. 53, 71, 83]. The author derives simultaneous confidence limits for all $\mathbf{u}'\boldsymbol{\tau}$ where $\mathbf{u}'$ belongs to the k ($\leqq m$) dimensional subspace spanned by

$$\mathbf{U}=\begin{pmatrix} u_{11} & \cdots & u_{1m} \\ u_{k1} & \cdots & u_{km} \end{pmatrix}$$

An illustrative example is given.

S. Kullback (Washington, D.C.)

7366:

Kruskal, William H. Ordinal measures of association. J. Amer. Statist. Assoc. **53** (1958), 814–861.

Rank measures of association for bivariate populations are discussed, with emphasis on the probabilistic and operational interpretations of their population values. The paper is mostly expository. There is a valuable historical outline of the subject.

W. Hoeffding (Chapel Hill, N.C.)

7367:

de Oliveira Carvalho, Pedro Egydio. On the distribution of the Kolmogorov-Smirnov D-statistic. Ann. Math. Statist. **30** (1959), 173–176.

Gnedenko and Korolyuk [Dokl. Akad. Nauk SSSR **80** (1951), 525–528; MR **13**, 570], and independently, Drion [Ann. Math. Statist. **23** (1952), 563–574; MR **14**, 488] have shown how the exact distribution of the Kolmogorov-Smirnov D-statistic can be obtained explicitly for finite sample size by solving a certain double-boundary random walk problem which, in turn, is solved by the principle of reflection. This principle is employed here in what is believed to be a new way to derive the distribution.

D. G. Chapman (Seattle, Wash.)

7368:

Walsh, John E. Comments on "The simplest signed-rank tests". J. Amer. Statist. Assoc. **54** (1959), 213–224.

Assume $X_1, \cdots, X_i, \cdots, X_n$ are independently distributed random variables, and the distribution function $F_i(x)$ of X_i is symmetric, i.e., $F_i(x)+F_i(-x)=1$. Form the distinct averages $(X_i+X_j)/2$ $(i, j=1, \cdots, n)$. Let Y_1 be the smallest of these averages, Y_2 the second smallest, etc, thus obtaining $Y_1 \leqq Y_2 \leqq \cdots \leqq Y_n(n+1)/2$. Details of test and confidence interval procedures based on the Y's are presented. The point is emphasized that these procedures are more inclusive than the Wilcoxon one-sample signed-rank procedures.

I. R. Savage (Minneapolis, Minn.)

7369:

Darwin, J. H. Note on a three-decision test for comparing two binomial populations. Biometrika **46** (1959), no. 1/2, 106–113.

If p_1 and p_2 are the probabilities of "success" with "treatments" T_1 and T_2, respectively, any experiment-pair involving T_1 and T_2 can result in a success-failure pattern the probabilities of which are $p_1(1-p_1)\equiv\pi_1$, $p_1p_2+(1-p_1)(1-p_2)\equiv\pi_2$ and $p_2(1-p_1)\equiv\pi_3$. Let these three cases be scored $+1$, 0 and -1, respectively, and write $y(n)$ for the total score after n experiment-pairs; $n=0, 1, 2, \cdots, y(0)=\frac{1}{2}(k+1)$. The author constructs three boundaries $y=0$, $y=k+1$ and $n=M$ analogous to those of Armitage's restricted sequential procedure [Biometrika **44** (1957), 9–26; MR **19**, 76] but depending on the parameter π_2 instead of ignoring it and its associated experiment-pairs. An exact-solution is obtained using matrix theory and good approximations are developed. It is shown that if T_1 is a "control", the lack of knowledge of π_2 may not be a disadvantage.

H. L. Seal (New York, N.Y.)

7370:

Stevens, W. L. Sampling without replacement with probability proportional to size. J. Roy. Statist. Soc. Ser. B. **20** (1958), 393–397.

The author considers the following sample plan for a stratified population: "Select, with replacement, a finite number of strata, each with probability proportional to its size. If the stratum i is chosen t_i times, select without replacement t_i units with equal probability from it" (quoted partly from the author's text). He gives an unbiased estimate of the population mean, calculates its variance and finds an unbiased estimate for this variance. [For similar work compare, for instance, F. Yates, and P. M. Grundy, J. Roy. Statist. Soc. Ser. B. **15** (1953), 253–261 or D. G. Horvitz and D. J. Thompson, J. Amer. Statist. Assoc. **47** (1952), 663–685; MR **14**, 777.]

L. Schmetterer (Berkeley, Calif.)

7371:

Hájek, Jaroslav. On the theory of ratio estimates. Apl. Mat. **3** (1958), 384–398. (Czech and Russian summaries)

"Estimated variances, yielded by large sample approach, are adjusted by a proportional regression approach; subsequently, under the assumption of normality, exact statements on confidence intervals are arrived at. The paper deals, too, with complex types of ratio estimates, as well as with modifications needed when stratification, multiple stages, or some special methods of first-stage sampling are present." (Author's summary) This paper yields a confidence interval for the sum of all values of a single variable over a finite population which is shown to be shorter than a confidence interval of E. C. Fieller [Quart. J. Pharm. **17** (1944), 117–123].

F. C. Andrews (Eugene, Ore.)

7372:

Davis, R. C. Optimum vs. correlation methods in tracking random signals in background noise. Quart. Appl. Math. **15** (1957), 123–138.

Suppose that processes $s(t)$, $n_1(t)$, $n_2(t)$, $-\infty<t<\infty$, are stationary, Gaussian, and real, and that s is independent of n_1 and n_2. Given an observation of $x_1(t)=s(t)+n_1(t)$ and $x_2(t)=s(t+\tau_0)+n_2(t)$ throughout the interval $0\leqq t\leqq T$, one seeks an estimate of the unknown parameter τ_0. The author obtains the Cramér-Rao lower bound for the variance of unbiased estimates of τ_0, and treats explicitly the finite time correlator estimate. (The quantity $\int_0^T x_1(t)\hat{x}_2(t)dt$, where $\hat{x}_2$ is the Hilbert transform of x_2, is a certain function of τ_0 which may be used to estimate τ_0.) The maximum likelihood estimate is discussed in certain limiting cases. Results appear in the form of infinite series whose terms are complicated functions of Fourier coefficients of the various auto- and cross-covariances relative to the interval $[-T, T]$. The author is not able to bring these series to closed form, and must rely on his engineering insight to extract useful information from them.

S. P. Lloyd (Murray Hill, N.J.)

7373:

Bochner, Salomon; and Kawata, Tatsuo. A limit theorem for the periodogram. Ann. Math. Statist. **29** (1958), 1198–1208.

Let $\mathscr{E}(t)$ be a real stationary process in the wide sense with mean zero and an absolutely continuous spectral distribution function. Further, let $\mathscr{E}(t)$ be stationary of fourth order so that

$$E\mathscr{E}(t)\mathscr{E}(t+u)\mathscr{E}(t+v)\mathscr{E}(t+w)=P(u, v, w)$$

exists and depends only on u, v, w. The authors introduce $Q(u, v, w)$, the difference between $P(u, v, w)$ and the corresponding fourth moment of a stationary Gaussian process. $Q(u, v, w)$ is very useful and was first introduced by T. Magness [J. Appl. Phys. **25** (1954), 1357–1365]. The random variable

$$S(T)=T^{-1}\left|\int_{-\infty}^{\infty}\mathscr{E}(t)M(t/T)e^{-i\xi t}dt\right|^2$$

which the authors call a generalized periodogram, is studied. Under appropriate conditions on $Q(u, v, w)$ and $M(x)$, the limiting covariance structure of $S(T)$ as $T\to\infty$ is obtained.

M. Rosenblatt (Bloomington, Ind.)

7374:

Siddiqui, M. M. Covariances of least-squares estimates when residuals are correlated. Ann. Math. Statist. **29** (1958), 1251–1256.

Expressions are obtained for the covariances of the least squares estimates of regression coefficients when the regression is either a linear or trigonometric regression. The residuals are assumed to be weakly stationary. Asymptotic expressions are obtained for the covariances when the residuals are serially correlated. The work is strongly motivated by the fact that the least squares estimates are asymptotically efficient in the class of linear unbiased estimates under rather broad conditions. Work on the asymptotic efficiency of least squares estimates is due to U. Grenander [Ann. Math. Statist. **25**

(1954), 252–272; MR 15, 973] and to U. Grenander jointly with the reviewer [Statistical analysis of stationary time series, John Wiley, New York, 1957; MR 18, 959].

M. Rosenblatt (Bloomington, Ind.)

7375:

Firescu, D. Sur les fonctions d'estimation des probabilités de passage d'une chaîne de Markoff. An. Univ. "C. I. Parhon" București. Ser. Ști. Nat. 7 (1958), no. 18, 9–18. (Romanian. Russian and French summaries)

Consider a simple homogeneous discontinuous Markoff chain with values $a_1, a_2, \cdots, a_m$ and transition probabilities p_{ij} ($i\in I_m, j\in I_m, I_m=(1, 2, \cdots, m)$). Let n_i be the number of appearances of the state a_i in a sequence of n experiences and n_{ij} the number of appearances of the pair a_i, a_j in this sequence.

Put

$$V_s^n=\|n_{ij_1}n_{ij_2}\cdots n_{ij_r}\|,\quad r<m,$$

assuming that

$$P[x_1=a_h]=p_{sh}.$$

Supposing the matrix $\|p_{hk}\|$ undecomposable, the author proves the following theorems: I: The distribution law of the random variable

$$W_s^n=\frac{V_s^n-M(V_s^n)}{\sqrt{n}},\ s\in I_m,$$

tends, for $n\to\infty$, to the normal law of r dimensions; II: The reduced distribution law of the random vector

$$\left\|\frac{n_{ij_1}}{n}\ \frac{n_{ij_2}}{n}\cdots\frac{n_{ij_r}}{n}\right\|$$

tends, for $n\to\infty$, to the normal law for r dimensions.

With the aid of these theorems it is proved that the estimation functions n_{ij_s}/n ($s=1, 2, \cdots, r$; $r<m$) of the unknown probabilities p_{ij_s} are Gaussian.

The functions become efficient only in the case, considered by Gh. Mihoc, when $r=m$.

In order to establish these results, the author applies properties of a linear operator of the type used by Onicescu and Mihoc in their studies concerning Markoff chains.

O. Onicescu (Bucarest)

7376:

Klein, L. R. The estimation of distributed lags. Econometrica 26 (1958), 553–565.

Let $y_t=\alpha\sum_{i=0}^{\infty}\lambda^i x_{t-i}+u_t$ and the residuals u_t follow a Markoff process: $u_t=\xi u_{t-1}+e_t$, where the e_t are not autocorrelated [L. M. Koijck, Distributed lags and investment analysis, North Holland, Amsterdam, 1954]. If $\xi=0$ the problem can be considered as one of weighted regression and α and λ can be estimated with the help of a solution of a quadratic equation. If ξ also has to be estimated an iterative method is suggested. It is shown in an appendix that the estimation methods are not full but limited information maximum likelihood methods. Not all the implied restrictions are taken into account.

G. Tintner (Lisbon)

7377:

White, John S. The limiting distribution of the serial correlation coefficient in the explosive case. Ann. Math. Statist. 29 (1958), 1188–1197.

The author considers the limiting distribution of the least-square estimate $\hat{\alpha}$ of the autoregression coefficient α of a first-order linear normal autoregression. He shows that the limiting distribution of $\hat{\alpha}$, suitably normalised, is Cauchy when $|\alpha|>1$, normal when $|\alpha|<1$, and of another form, for which the characteristic function is given, when $|\alpha|=1$.

P. Whittle (Cambridge, England)

NUMERICAL METHODS

See also 7407, 7460, 7549, 7589, 7590, 7595.

7378:

★Moore, P. G. Principles of statistical techniques: a first course, from the beginnings, for schools and universities. Cambridge University Press, New York, 1958. viii+239 pp. $3.75.

7379:

Wensley, J. H. A class of non-analytical iterative processes. Comput. J. 1 (1959), 163–167.

The author is concerned with those iterative techniques where the correction depends not upon the error at a prior step but upon how many times the process has been done. Thus, a computer division or square root is performed as many times as there are binary or decimal places. He gives a considerable number of flow diagrams for solving functional equations by such methods.

H. H. Goldstine (Yorktown Heights, N.Y.)

7380:

Sahelov, G. S.; and Mertvecova, M. A. Convergence of certain iterative processes. Advancement in Math. 3 (1957), 341–374. (Chinese)

7381:

★Ostrowski, Alexander M. On trends and problems in numerical approximation. On numerical approximation. Proceedings of a Symposium, Madison, April 21–23, 1958, pp. 3–10. Edited by R. E. Langer. Publication no. 1 of the Mathematics Research Center, U.S. Army, the University of Wisconsin. The University of Wisconsin Press, Madison, 1959. x+462 pp. (1 insert) $4.50.

An entertaining and informative survey, beginning with a discussion of the place of rigor in "the science of computation", touching upon the application of statistics to the assessment of errors, and tracing briefly developments in interpolation and in methods of successive approximation.

A. S. Householder (Oak Ridge, Tenn.)

7382:

★Todd, John. Special polynomials in numerical analysis. On numerical approximation. Proceedings of a Symposium, Madison, April 21–23, 1958, pp. 423–446. Edited by R. E. Langer. Publication no. 1 of the Mathematics Research Center, U. S. Army, the University of Wisconsin. The University of Wisconsin Press, Madison, 1959. x+462 pp. (1 insert) $4.50.

The author defines the place of special polynomials in numerical analysis. He outlines such a course which he divides into the following chapters: The Weierstrass Theorem; The Tchebycheff Polynomials; The Theorems of Markoff; Orthogonality; Approximate Quadrature; and Interpolation Processes. In places which are likely to be less familiar, the outline is presented in some detail including the proof of theorems. *S. Kulik* (Logan, Utah)

7383:

Salzer, Herbert E. Formulae for hyperosculatory interpolation, direct and inverse. Quart. J. Mech. Appl. Math. 12 (1959), 100–110.

Ordinary (Lagrange) interpolation makes use of the values of a function, $f(x_i)$, at certain abscissas x_i. Osculatory interpolation uses both the functional and derivative values, $f(x_i)$ and $f'(x_i)$. Hyperosculatory interpolation takes advantage of $f(x_i)$, $f'(x_i)$, and $f''(x_i)$. The

author presents constants which facilitate the use of equidistant hyperosculatory interpolation formulas, both direct and inverse. The utility of this mode of interpolation is pointed out for the case of solutions of second order differential equations which are tabulated along with their derivatives and for certain problems of rocket flight. *P. Davis* (Washington, D.C.)

7384:

Hornecker, Georges. Évaluation approchée de la meilleure approximation polynomiale d'ordre n de $f(x)$ sur un segment fini $[a, b]$. Chiffres 1 (1958), 157–169.

Soit $f(x)$ continue dans $[-1, +1]$, comme évaluation de sa meilleure approximation par un polynome de degré n, on prend celle donnée par le polynome de la meilleure approximation aux points:

$$x_k+\delta x_k,\ \text{avec}\ x_k=\cos k\frac{\pi}{n+1},\ \delta x_k=\frac{2}{n+1}(1-x_k^2)\ \frac{c_{n+2}}{c_{n+1}}$$

$$+x_k\left(2\ \frac{c_{n+3}}{c_{n+1}}-\frac{n+4}{n+1}\frac{c_{n+2}^2}{c_{n+1}^2}\right)(k=0, 1, \cdots, n+1)$$

où c_n désigne le coefficient du développement de $f(x)$ en série de polynomes de Tchebitcheff. Examen des cas particuliers où $f(x)$ est paire ou impaire. Applications numériques diverses. Meilleure approximation de $(a+x)^{-1}$ $(a>0, 0\leq x\leq 1)$. *J. Favard* (Paris)

7385:

Kosambi, D. D. The method of least squares. Advancement in Math. 3 (1957), 485–491. (Chinese)

7386:

Herzel, Amato. Sull'interpolazione delle funzioni continue coi metodi di Bessel e Everett. Fac. Sci. Statist. Demogr. Attuar. Ist. Statist. Ist. Calcolo Probab. No. 43, 29 pp. (1958).

A generalization of throwback.

A. S. Householder (Oak Ridge, Tenn.)

7387:

Householder, Alston S.; and Bauer, Friedrich L. On certain methods for expanding the characteristic polynomial. Numer. Math. 1 (1959), 29–37.

The authors show in detail how a number of methods for finding the characteristic polynomial of a matrix are in essence rather direct extensions of Krylov's method. Previously, one of the authors, Bauer, had showed that the well-known method of Danilevskii was such an extension. Here, the authors show very elegantly how a considerable number of well-known methods are of the same species. *H. H. Goldstine* (Yorktown Heights, N.Y.)

7388:

★Eybert, P. Sur un procédé de capture des racines des équations algébriques. Actes des colloques de calcul numérique, Caen, 1955; Dijon, 1956; pp. 121–126. Publ. Sci. Tech. Ministère de l'Air, Notes Tech. no. 77, Paris, 1958. vi+144 pp. 2105 francs.

Le procédé envisagé consiste à former pour l'équation $f(x)=0$ degré p, les expressions:

$$\sum (x-x_i)\psi,\ \sum_i (x-x_i)\psi(x_i)$$

$$\sum_{ij} (x-x_i)(x-x_j)\psi(x_i)\psi(x_j),\ \text{etc.}$$

On choisit la fonction arbitraire $\psi(x_i)$ grande au voisinage des racines que l'on veut capter.

Le procédé peut être appliqué à la recherche des racines voisines d'un nombre donné, ou voisines d'un cercle, ou voisines d'un segment rectiligne.

J. Kuntzmann (Grenoble)

7389:

Bahvalov, S. V. An inverse geodesic problem. Moskov. Oblast. Pedagog. Inst. Uč. Zap. 57 (1957), 143–151. (Russian)

On donne une méthode numérique (qui converge rapidement) pour obtenir la longueur de l'arc géodésique aux extrémités données d'un ellipsoïde de révolution.

A. Švec (Prague)

7390:

Bajcsay, P.; und Lovass-Nagy, V. Ein Iterationsverfahren zur näherungsweisen Lösung von Matrizendifferentialgleichungen. Z. Angew. Math. Mech. 39 (1959), 8–13. (English, French and Russian summaries)

A method is proposed for solving the system of differential equations $(d/dt)y=Ky$, $y(0)=y_0$ ($y(t)$, y_0 column vectors of dimension n and K a constant matrix) without transforming K to canonical form. K is decomposed into a sum $A+B$ with the matrix A having a known canonical representation (e.g., A a diagonal matrix), and the following iteration is set up: $(d/dt)y_1=Ay_1$, $(d/dt)y_m=Ay_m+By_{m-1}$ $(m=2, 3, \cdots)$, $y_m(0)=y_0$ $(m\geq 1)$. An explicit expression

$$y_m(t)=\sum_{\nu=1}^{n}\beta_{m\nu}(t)\exp(\lambda_\nu t)u_\nu$$

is obtained, where λ_ν are the eigenvalues of A, u_ν corresponding eigenvectors, and $\beta_{m\nu}(t)$ scalar functions (too lengthy to be quoted here) formed with the help of the matrix B and the right and left-hand eigenvectors of A. An analogous method is stated for inhomogeneous systems, and convergence is proved as $m\to\infty$.

Walter Gautschi (Washington, D.C.)

7391:

★Dahlquist, Germund. Stability and error bounds in the numerical integration of ordinary differential equations. Inaugural dissertation, University of Stockholm, Almqvist & Wiksells Boktryckeri AB, Uppsala, 1958. 87 pp.

Same as Kungl. Tekn. Högsk. Handl. Stockholm. No. 133 (1959) [MR 21 #1706].

7392:

Schröder, Johann. Fehlerabschätzungen bei gewöhnlichen und partiellen Differentialgleichungen. Arch. Rational Mech. Anal. 2 (1958/59), 367–392.

Boundary value problems of the form (1) $M[u]=f(x, u)$ in $\mathfrak{B}$, $U_i[u]=\gamma_i$ $(i=1, 2, \cdots)$ on Γ are considered, where $\mathfrak{B}$ is a bounded region of the Euclidean space, Γ its boundary, and M, U_i are linear homogeneous differential operators. Denoting a solution of (1) by u^*, the author seeks to estimate the errors u_n-u^* of approximations u_n obtained by the iteration procedure

$$(2)\qquad M[u_{n+1}]=f(x, u_n),\ U_i[u_{n+1}]=\gamma_i\ (n=0, 1, 2, \cdots).$$

It is assumed, among other things, that there exists a Green's function $G(x, \xi)$ for problem (1). This makes it possible to study in place of (2) an equivalent iteration of the form

$$u_{n+1}(x)=g(x)+\int_{\mathfrak{B}} G(x, \xi)f(\xi, u_n(\xi))d\xi.$$

Earlier results of the author [same Arch. 1 (1957), 154–180; MR 20 #2072] relative to iterations $u_{n+1}=Tu_n$ in a complete metric space are then applicable, which establish error bounds by means of a "comparison iteration" $\rho_{n+1}=\bar{T}\rho_n$. In the present case, and in its simplest version, the comparison iteration takes the form

$$\rho_0(x)\equiv 0,\ \rho_{n+1}(x)=\sigma_1(x)+\int_{\mathfrak{B}}|G(x,\xi)|\bar{f}(\xi,\rho_n(\xi))d\xi,$$

where σ_1 bounds $|u_1-u_0|$ and $\bar{f}(x,y)$ is a "majorant", in a specified sense, of the function $f(x,y)$. The main result states that if $\sigma(x)$ satisfies

$$\sigma(x)\geq\int_{\mathfrak{B}}|G(x,\xi)|+\bar{f}(\xi,\sigma(\xi)+\sigma_1(\xi))d\xi,$$

then $|u_1-u^*|\leq\sigma$ and, more generally, $|u_n-u^*|\leq\sigma+\sigma_1-\rho_n$. If $G(x,\xi)\geq 0$, then σ can also be taken as a solution of the "comparison problem" $M[\sigma]\geq\bar{f}(x,\sigma+\sigma_1)$, $U_i[\sigma]=0$. Several examples of possible majorants $\bar{f}(x,y)$ are exhibited and their appropriate choice is discussed.

The method is illustrated for certain classes of boundary (and initial) value problems, both for ordinary and partial differential equations, and four numerical examples are worked out in detail. *Walter Gautschi* (Washington, D.C.)

7393:

Sokolov, Yu. D. Sur la résolution approchée des équations intégrales linéaires du type de Volterra. Ukrain. Mat. Ž. 10 (1958), no. 2, 193–208. (Russian. French summary)

The author devises a method of "averaging functional corrections" for solution of Volterra integral equations of the second kind as an extension of previous work on a similar technique for Fredholm equations. For the equation

$$y(x)=\varphi(x)+\int_a^x K(x,\xi)y(\xi)d\xi$$

the author proposes the iteration process

$$y_0=0;\ y_n(x)=\varphi(x)+\int_a^x K(x,\xi)[y_{n-1}(\xi)+\alpha_n]d\xi,$$

$$\alpha_n=h^{-1}\int_a^{a+h}[y_n(x)-y_{n-1}(x)]dx.$$

Various criteria for convergence as a function of h are given. The author applies the theory to numerous examples, including multi-dimensional problems such as the solution of a hyperbolic initial-value second order partial differential equation; and shows numerically in these cases at least that the method produces more rapid convergence than the classical Cauchy-Picard process where the α_i above are set equal to zero. He notes also that the method can be extended to the more general multi-dimensional case. *J. W. Carr, III* (Chapel Hill, N.C.)

7394:

Nikolaev, P. V. On projectivity of nomographic representations of equations. Mat. Sb. N.S. 45(87) (1958), 369–396. (Russian)

7395:

Morita, Katuhiko. Complex concircular chart. Kagaku 29 (1959), 148–149. (Japanese. English summary)

We treat here the functional relation

$$\frac{f_1(z_1)-f_3(z_3)}{f_1(z_1)-f_4(z_4)}:\frac{f_2(z_2)-f_3(z_3)}{f_2(z_2)-f_4(z_4)}=a\ \ (a\text{ real const.}),$$

where $z_j=x_j+iy_j$, $i^2=-1$, $j=1,2,3,4$. If we put

$$w_j=f_j(z_j)=f_j(x_j+iy_j)=u_j(x_j,y_j)+iv_j(x_j,y_j),$$

we have the following four expressions as the scale equations of our "Complex concircular chart"

$$(z_j):\ u_j=u_j(x_j,y_j),\quad v_j=v_j(x_j,y_j)\ (j=1,2,3,4).$$

Representing graphically the above scale equations on the Gaussian plane uov, we obtain the four systems of curvilinear nets, and an intersection point of a certain Apollonius' circle in regard to $\overline{Z_1Z_2}$ and a circle K, passing through the three points corresponding to the three given z_j, is the required point which gives the required value of the fourth z. *Author's summary*

7396:

★Gauss, F. G. Fünfstellige vollständige logarithmische und trigonometrische Tafeln. Herausgegeben von Dr.-Ing. H. H. Gobbin. 391.–400. Aufl. Verlag Konrad Wittwer, Stuttgart, 1958. xxxii+184 pp. DM 4.80.

The mathematical tables in this volume include: $\log x$ for $x=1(1)10000$, 5D, and $x=10000(1)11000$, 7D; $\log\sin x$, $\log\cos x$, $\log\tan x$ and $\log\cot x$ for $x=0°(1'')1°(10'')8°$, and $x=0°(1')90°$, 5D; $\ln x$, for $x=0(1)1100$, 5D; $\sin x$, $\cos x$, $\tan x$, $\cot x$, for $x=0°(10')90°$, 4D; a table of addition and subtraction logarithms; a table of chord-length and similar quantities for the unit circle; and miscellaneous tables of constants involving π, e, etc.

The tables are mostly linearly interpolatable, and are provided with proportional parts and first differences where necessary. Some indication of an extra figure is given by means of a dot or bar on the last figure.

There are also some forty pages of tables of physical quantities, including measurements of distance, position and time at various places on earth; relations between various systems of units of mass, electricity, heat and atmospheric pressure; facts about the chemical elements and various physical properties of several substances; and some astronomical tables relating to the sun, moon, earth, planets and stars.

A final section illustrates the use of the various mathematical tables. The whole forms an interesting collection of information, more useful to the chemist and physicist than to the mathematician. *L. Fox* (Oxford)

7397:

Cohn, Harvey. A computation of some bi-quadratic class numbers. Math. Tables Aids Comput. 12 (1958), 213–217.

A table is given of the class numbers of the biquadratic number fields generated by $\sqrt{5}$ and $(-\mu)^{\frac{1}{2}}$ where $\mu=\frac{1}{2}(a+b\sqrt{5})$ and $0<5b<a<100$. This table was computed on an I.B.M. 650 electronic computer.

C. B. Haselgrove (Manchester)

7398:

★Vogel, Alfred. Vierstellige Funktionentafeln. Verlag Konrad Wittwer, Stuttgart, 1958. viii+157 pp. DM 6.80. (2 inserts: Differenzentafeln, 2 pp.; Mathematische Formelsammlung, 24 pp.; Formelsammlung allein DM 1.40)

Four-place tables of elementary functions for use in high schools. *Walter Gautschi* (Washington, D.C.)

COMPUTING MACHINES

See also 7397, 7599, 7600.

MECHANICS OF PARTICLES AND SYSTEMS

See also 7134, 7457, 7521, 7579.

7399:

Miron, R. Le problème de la géométrisation des systèmes mécaniques non holonomes. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **7** (1956), no. 1, 15–49. (Romanian. Russian and French summaries)

The problem of the geometrization of anholonomic mechanical systems with constraints independent of time has been solved by E. Cartan [Atti del Congresso Internat. dei Matem. Bologna, 1928, vol. 4, pp. 253–261, Zanichelli, Bologne, 1931], G. Vranceanu [Mémor. Sci. Math. no. 76 (1936), 1–70] and M. Haimovici [Acad. R.P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Ser. I **5** (1954), no. 3–4, 49–84; MR **16**, 1167] with the hypothesis that the first derived system of the Pfaff system which expresses the kinematical constraints is zero. In the present paper the author studies the same problem assuming that the Pfaff system has r derived systems, i.e., the $r+1$ derived system is zero. This requires the consideration of the so-called 'intrinsic' group G_r which leaves invariant the Pfaff system and its r derived systems, and the subgroup of G_r which leaves invariant the equations of motion of the mechanical system.

This subgroup is obtained by adding to G_r a system of linear and homogeneous equations in some of the coefficients of G_r. Two cases are now possible: 1) Either this system has only the trivial solution zero, in which case the subgroup is denoted by Γ_r^1, or 2) it has a non-trivial solution, in which case the subgroup is denoted by Γ_r^2.

Finally, the author considers the equivalence problem of mechanical systems in case 1), leaving case 2) as the subject of a subsequent paper.

R. Blum (Saskatoon, Sask.)

7400a:

Miron, R. Sur la géométrie intrinsèque des variétés non holonomes. Com. Acad. R. P. Romîne **7** (1957), 5–11. (Romanian. Russian and French summaries)

7400b:

Miron, R. Sur la géométrie intrinsèque des variétés non-holonomes. II. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **8** (1957), no. 1, 49–73. (Romanian. Russian and French summaries)

7400c:

Miron, R. Sur la géométrie intrinsèque des variétés non holonomes. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **2** (1956), 85–103. (Romanian. Russian and French summaries)

Based on the results of #7399, reviewed above, the author studies the geometrization of Pfaff systems which have r derived systems, when these are subjected to transformations of the 'intrinsic' group. In the first two papers certain assumptions concerning the rank of matrices are made. In the third paper these assumptions are renounced. As a result there appear in the derived systems a number of invariant subsystems. It is shown that in this case the same methods can be applied as before.

R. Blum (Saskatoon, Sask.)

7401:

Braier, Alfred. L'équation différentielle des trajectoires décrites par un point matériel dans un champ de forces. Bul. Inst. Politehn. Iaşi (N.S.) **4(8)** (1958), 103–112. (Russian and Romanian summaries)

On sait comment, lorsqu'un champ de forces dérive d'un potentiel, l'élimination du temps t se fait au moyen du principe de Maupertuis. Cette méthode suggère d'utiliser, dans le cas général d'un champ de forces fonction de la position, une expression du carré de la vitesse sous une forme indépendante du paramètre u choisi: par exemple, dans le cas des coordonnées cartésiennes rectangulaires x, y, les composantes X, Y de la force étant supposées connues en fonction des coordonnées, une telle expression est

$$\frac{Y\dot{x}-X\dot{y}}{m(\ddot{y}\dot{x}-\ddot{x}\dot{y})}(\dot{x}^2+\dot{y}^2)$$

où les points désignent des dérivations par rapport à u. D'où l'équation de la trajectoire

$$2(X+Yy')=\frac{d}{dx}\left(\frac{(Y-Xy')(1+y'^2)}{y''}\right)$$

où y', y'' désignant les dérivées première, seconde de y par rapport à x. L'auteur donne les formules générales correspondantes dans le cas de coordonnées curvilignes orthogonales quelconques, soit dans le plan, soit dans l'espace.

M. Janet (Paris)

7402:

Minorsky, Nicolas. Sur l'action asynchrone. C. R. Acad. Sci. Paris **248** (1959), 631–633.

In this note the author reviews some of the known facts concerning the dependence of the properties of the solutions of the equation $\ddot{x}+x+\mu f(x,\dot{x})=e\sin\omega t$ on the value of the parameter ω. The stroboscopic method is used, in the case in which $f(x,\dot{x})=(\gamma x^2-\alpha)\dot{x}$, to obtain formulae which are interpreted as showing that a large value of ω results in the suppression of the autoperiodic oscillation.

L. A. MacColl (New York, N.Y.)

7403:

Sapa, V. A. Equations of the motion of a system of material points of variable mass in generalized coordinates. Canonical equations. Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh. no. **6(10)** (1957), 60–81. (Russian. Kazah summary)

In this paper A. A. Kosmodem'yanskiĭ's hypotheses under which the Lagrangian equations of the second kind are deduced in generalized coordinates [Moskov. Gos. Univ. Uč. Zap. 154, Meh. **4** (1951), 73–180; MR **14**, 1025] are extended. Assuming a material in which l masses are constant, k masses change by ejecting particles and r masses change by simultaneously ejecting and adding particles, the Lagrangian equations of the second kind for holonomic systems and Routh's and Appel's equations for non-holonomic systems are deduced. Several particular cases of these equations are discussed in detail. It is pointed out that Kosmodem'yanskiĭ's equations can be deduced from the Lagrangian equations of motion. Further, it is shown that for the case where the external forces are conservative and where the absolute velocities of the ejected and added particles are equal to zero, the Lagrangian equations of motion of the system with variable masses are formally the same as in the case of a conservative system with constant masses.

The method of Lagrangian equations is illustrated by three examples: the motion of a rotor of a reaction gas

turbine; Segner's turbine submerged in ejecting water; and the differential equations of motion of a water turbine.

At the end of the paper the canonical equations of motion are derived also under the same hypotheses. Seven particular cases are discussed. *D. P. Rašković* (Belgrade)

STATISTICAL THERMODYNAMICS AND MECHANICS

See also 7337, 7439, 7476.

7404:

Maradudin, A. A.; Mazur, P.; Montroll, E. W.; and Weiss, G. H. Remarks on the vibrations of diatomic lattices. Rev. Mod. Phys. **30** (1958), 175–196.

This is mainly a review of selected topics in the theory of crystal vibrations, restricted primarily to aspects which the authors themselves have helped develop. The first section contains an interesting review of the history of the analysis of vibration spectra, beginning with related work of Lord Kelvin and Lord Rayleigh and ending with the recent attempts to analyse singularities. This is followed by a derivation of the exact spectrum for an ordered diatomic cubic lattice with nearest neighbor forces only. The largest part of the paper, however, deals with various methods for approximating the spectrum of disordered binary alloys forming a cubic lattice and includes a discussion of (1) dilute mixtures, (2) slightly disordered structures, and (3) completely disordered arrangements.

G. Newell (Stockholm)

7405:

Weiss, George; and Maradudin, Alexei. Thermodynamic properties of a disordered lattice. Phys. and Chem. Solids **7** (1958), 327–344.

7406:

Vladimirov, V. S. Numerical solution of the kinetic equation for the sphere. Vyčisl. Mat. 3 (1958), 3–33. (Russian)

The author considers the linearized integro-differential equation of Boltzmann associated with a neutron transport process for a spherically symmetric reactor with stationary behavior over time. Analytic properties of the one-group and multi-group solutions are derived, and numerical results are obtained, by finite difference techniques, for the one-group case.

R. Bellman (Santa Monica, Calif.)

7407:

Birkhoff, Garrett; and Varga, Richard S. Reactor criticality and nonnegative matrices. J. Soc. Indust. Appl. Math. **6** (1958), 354–377.

By the use of the Perron-Frobenius theory of nonnegative matrices the authors develop a rigorous mathematical basis for the concept of criticality and for certain computational procedures involved in criticality determination. Particular attention is given to the problem of determining the neutron flux density in a heterogeneous reactor. The usual multigroup equations are

$$\begin{aligned} \frac{\partial\phi_i}{\partial t} &= v_i\sum_k \frac{\partial}{\partial x_k}\left(D_i\frac{\partial\phi_i}{\partial x_i}\right)-\Sigma_i'\,\phi_i' \\ (1)\qquad &\qquad +\Sigma_{i-1}\,\phi_{i-1}\ (i=2,\cdots,n), \\ \frac{\partial\phi_1}{\partial t} &= v_i\sum_k \frac{\partial}{\partial x_k}\left(D_1\frac{\partial\phi_1}{\partial x_k}\right)-\Sigma_1'\,\phi_1+\nu\Sigma_n\,\phi_n, \end{aligned}$$

where ϕ_i is the neutron flux density corresponding to the ith lethargy group with velocity v_i, D_i is the diffusion length, Σ_i the total cross section, and Σ_i' is the slowing-down cross section of the group ($\Sigma_i'<\Sigma_i$). The parameter ν represents the fission yield. In critical flux calculations the time derivatives are set equal to zero. Replacing the space derivatives by appropriate difference quotients one obtains

$$(2)\qquad \begin{aligned} &A_k\phi_k=B_k\phi_{k-1}\ (k=2,\cdots,n), \\ &A_1\phi_1=\nu B_1\phi_n, \end{aligned}$$

where the A_k and B_k are square matrices and the ϕ_k are column matrices. Because of the assumptions on the coefficients appearing in (1) the matrices A_k are non-singular. Moreover, the A_k^{-1} are positive matrices (having all positive elements), while the matrices $A_k^{-1}B_k$ are non-negative (having all non-negative elements).

A common computational procedure involves assuming $\phi_n(0)$ and determining $\phi_n(r+1)$, $r=0, 1, 2, \cdots$, from $\phi_n(r)$ by successively solving the sets of linear equations $A_1\phi_1(r+1)=\nu B_1\phi_n(r)$, $A_2\phi_2(r+1)=B_2\phi_1(r+1)$, $\cdots$, $A_n\phi_n(r+1)=B_n\phi_{n-1}(r+1)$ for $\phi_1(r+1)$, $\phi_2(r+1)$, $\cdots$, $\phi_n(r+1)$, respectively. Each set of equations is usually solved by an iterative process, each iteration of which is known as an inner iteration. The process of going from $\phi_n(r)$ to $\phi_n(r+1)$ is known as an outer iteration. The authors show that the product matrix $T=\nu(A_n^{-1}B_n\cdots A_1^{-1}B_1)$ has a positive eigenvector (with all positive components) which is unique to within a multiplicative factor. Moreover, it is shown that the iterative process converges to this vector.

More generally, the authors consider multiplicative processes, both discrete processes defined by equations of the form $N_i(r+1)=\sum_{j=1}^m p_{i,j}N_j(r)$ $(i=1, 2, \cdots, m)$, where $P=(p_{i,j})$ is a non-negative matrix, and also continuous processes defined by $(dN_i/dt)=\sum_{j=1}^m q_{i,j}N_j(t)$, where $Q=(q_{i,j})$ is essentially non-negative, i.e., $q_{i,j}\geqq 0$ for $i\neq j$. Multiplicative processes are generalizations of Markoff processes, where $\sum_{i=1}^m p_{i,j}=1$ $(j=1, 2, \cdots, m)$ in the discrete case, and where $\sum_{i=1}^m q_{i,j}=0$ $(j=1, 2, \cdots, m)$ in the continuous case. In the analysis of multiplicative processes the authors consider a number of types of matrices including non-negative, essentially non-negative, positive, essentially positive, irreducible, and semi-irreducible matrices. Matrices which are non-negative, essentially non-negative, positive, essentially positive, or semi-irreducible and non-negative have a non-negative eigenvector. Cyclic matrices are also considered. A cycle of length s is a sequence of non-zero elements a_{i,j_1}, a_{j_1,j_2}, $\cdots$, $a_{j_{s-1},i}$. A matrix is cyclic of index $k>1$ if the greatest common divisor of the lengths of its cycles is k. If $k=1$ the matrix is said to be primitive. Stieltjes matrices, which are symmetric, irreducible matrices with non-positive off-diagonal elements, are also considered.

By the use of special properties of the eigenvectors and eigenvalues of the types of matrices involved the authors obtain a number of results besides those already mentioned for reactor criticality problems. Results are obtained concerning the rates of convergence of various inner

iteration procedures including the successive over-relaxation method. The authors show that for symmetric essentially positive matrices, the spectral norm $L(P)$ of the matrix P (the maximum of the absolute values of its eigenvalues) is a convex function of P. The authors also construct transition matrices, relating Gauss iteration to sub-criticality, and treat thermal up-scattering. They conclude by studying cases in which the matrix

$$Q=\begin{vmatrix} -v_1A_1 & 0 & \cdots & 0 & \nu B_1 \\ v_2B_2 & -v_2A_2 & \cdots & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & v_nB_n & -v_nA_n \end{vmatrix}$$

has complex eigenvalues. It is shown that if $n\geqq4$ and if the smallest value of $v_i\Sigma_i'$ exceeds one-fifth of the fourth smallest $v_i\Sigma_i'$, then Q has at least one complex eigenvalue. Considerations of this kind are important from the computational standpoint, since if the eigenvalues are all real, then a method based on the use of Chebysheff polynomials is available for appreciably accelerating the convergence of the outer iteration process.

D. M. Young, Jr. (Austin, Tex.)

7408:

van Kampen, N. G. Derivation of the phenomenological equations from the master equation. I. Even variables only. Physica 23 (1957), 707–719.

The phase space of a thermodynamic system is divided into cells according to the values a_r of a set of macroscopic variables $A^{(r)}$. It is then possible to write a master equation for the evolution of the distribution function $P(a)$ in terms of the probability $W(a, a')$ of the system making a transition from state a to state a'. Through use of the principle of detailed balance, together with the assumption that $W(a, a')$ is small except when a_r and a_r' are almost equal, the master equation can be written as a Fokker-Planck equation describing a diffusion in a-space. By taking suitable averages of this equation and by limiting the discussion to small fluctuations about the equilibrium state, the usual phenomenological equations of irreversible thermodynamics can be derived. The coefficients are expressed in terms of the $W(a, a')$ and satisfy the Onsager reciprocal relations. It is shown that the mean square deviation of a_r decays exponentially to its equilibrium value at the same rate as the mean value of a_r itself. This last result is also obtained by introducing random force terms into the phenomenological relations and applying Brownian motion theory.

S. Prager (Brussels)

7409:

van Kampen, N. G. Derivation of the phenomenological equations from the master equation. II. Even and odd variables. Physica 23 (1957), 816–824.

The results of the preceding article are extended to the case where some of the variables $A^{(r)}$ are odd, i.e., change sign when the motion of the system is reversed. The statement of the principle of detailed balance needs to be slightly modified for this purpose, and the resulting Fokker-Planck equation contains a drift term in addition to the diffusion term previously obtained. For small deviations from equilibrium, however, the usual phenomenological equations can still be derived, and the Onsager reciprocal relations remain valid. The fluctuations in the $A^{(r)}$ are also discussed and applied to derive Nyquist's formula. *S. Prager* (Brussels)

ELASTICITY, PLASTICITY

See also 7030, 7582.

7410:

Truesdell, C. The rational mechanics of materials—past, present, future. Appl. Mech. Rev. 12 (1959), 75–80.

This review traces the history of continuum mechanics from the early ideas of the 13th century to the present day. Included in the discussion are the contributions of the Bernoullis, Euler, Saint-Venant, Newton and other leading workers of the 17th and 18th centuries in classical elasticity and hydrodynamics. There follows an account of Cauchy's work on the theory of strain and the history of Saint Venant's principle, culminating with Sternberg's work on the latter in 1953. Theories of rods and shells are also discussed. The more recent developments considered include the work on large elastic deformations and non-linear continuum mechanics of Rivlin and others and the relation of this work to statistical theories.

J. E. Adkins (Nottingham)

7411:

Bernstein, B.; and Ericksen, J. L. Work functions in hypo-elasticity. Arch. Rational Mech. Anal. 1 (1958), 396–409.

The work done by the stresses in deforming a perfectly elastic body depends only on the initial and final state of the body. In the theory of hypo-elasticity proposed by C. Truesdell [J. Rational Mech. Anal. 4 (1955), 83–133; MR 16, 880], however, the work may be a functional of the entire history of the stress. The authors study two special types of hypo-elastic materials in which the work function reduces to a function of the initial and final stress only.

Materials of type I are characterized by the existence of an initial stress t_{ij}^0 such that the work done is non-negative for every deformation starting from a state with stress t_{ij}^0. For such materials, the authors show that $t_{ij}^0=0$, and that the work per unit volume in the final state is a function $\gamma(t_{ij})$ of the final stress t_{ij} only, provided that the deformation starts from an unstressed state.

Materials of type II are characterized by the property that the work done is non-negative for all deformations in which the initial and final stresses coincide. For such materials, the work per unit initial volume, for each initial stress t_{ij}^0, turns out to be a function $\Theta(t_{ij})$ of the final stress t_{ij} only. If the initial and final stresses coincide, the work is actually zero. *W. Noll* (Pittsburgh, Pa.)

7412:

Tolokonnikov, L. A. Plane deformation of incompressible material. Soviet Physics. Dokl. 119 (3) (1958), 453–455 (1124–1126 Dokl. Akad. Nauk SSSR).

The author presents an analysis of the theory of finite elastic plane strain in terms of principal extension ratios and rotations at each point of the material. He defines the 'intensity of deformation' ($\mathfrak{I}$) for an incompressible material as the logarithm of one of the principal strains and assumes that the octahedral tangential stress τ is a single valued function of $\mathfrak{I}$. For the particular law $\tau\propto\tanh\mathfrak{I}$ ascribed to non-ferrous metals the governing differential equation and its solution are expressed as power series in a small parameter. The claim that the magnitude of the strains is not limited by this procedure does not seem entirely consistent. It is not clear that the materials examined necessarily possess a strain energy function.

J. E. Adkins (Nottingham)

7413:

Kil'čevs'kiĭ, M. O. H. Hertz's investigations of the contact problem and some stages of their further development. Akad. Nauk Ukrain. RSR. Prikl. Meh. **4** (1958), 121–129. (Ukrainian. Russian and English summaries)

"The paper deals with the solution of the static and dynamic contact problems of elasticity theory in accordance with Hertz's method and the generalizations of Hertz's solutions developed by Soviet scientists. The application of Gauss's principle to the solution of the contact problem is noted." (Author's summary)

H. G. Baerwald (Albuquerque, N.M.)

7414:

Kalandiya, A. I. A plane problem of Hertz type on compression of cylindrical bodies. Soobšč. Akad. Nauk Gruzin. SSR **21** (1958), 3–10. (Russian)

An elastic disk of one material is initially situated eccentrically in a circular hole in an infinite plane sheet of another material. The difference in radii is small. Compressive forces distributed along half of the circumference of the hole produce contact with the disk. Beginning with the boundary conditions due to the deformation, the author uses the method of complex stress functions to deduce an integral equation satisfied by the stresses in the disk. *R. N. Goss* (San Diego, Calif.)

7415:

Solomon, L.; and Drăghicescu, D. Sur l'emploi des transformations conformes dans le problème plan de l'élasticité pour des domaines doublement connexes. Acad. R. P. Romîne. Stud. Cerc. Mec. Apl. **8** (1957), 1115–1132. (Romanian. Russian and French summaries)

As the results of the present paper are properly summarized by the author, we reproduce his summary below.

"Dans ce travail, on emploie pour la solution du problème plan de l'élasticité pour les domaines bornés doublement-connexes, un algorithme alternant, ressemblant à celui de Schwarz généralisé.

Cet algorithme conduit à la construction des fonctions de Moushélichvilli par approximations successives. Les approximations impaires sont les solutions du problème pour le domaine simplement-connexe, intérieur à la frontière extérieure, avec les conditions aux limites corrigées par les approximations paires précédentes; les approximations paires sont les solutions pour le domaine infini simplement-connexe, extérieur à la frontière intérieure, avec les conditions à la limite corrigées par les approximations impaires précédentes.

Comme conditions à la limite, nécessaires pour le calcul des deux premières approximations, on a pris les données du problème.

Pour illustrer le procédé, on a calculé les trois premières approximations du problème élastique plan pour le domaine compris entre deux carrés concentriques ayant les cotés 4, respectivement 2.

On peut appliquer sans restrictions le procédé même pour le domaine compris entre deux rectangles et il est susceptible d'être appliqué pour les domaines à connexion multiple."

The author's solutions are presented in a form which are amenable to direct numerical calculations. The method of Schwarz is taken from Kantorovič and Krylov's important book [Približennye metody vysšego analiza, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950; MR **13**, 77]. *K. Bhagwandin* (Oslo)

7416:

Teodorescu, Petre P. About some spatial problems of the theory of elasticity. Acad. R. P. Romîne. Stud. Cerc. Mec. Apl. **8** (1957), 1101–1113. (Romanian. Russian and English summaries)

The author obtains solutions of three-dimensional stress distribution problems in terms of harmonic and biharmonic polynomials, as well as Fourier-series approximations, when a body is bounded by plane parallel surfaces with respect to the coordinate surfaces (plane).

K. Bhagwandin (Oslo)

7417:

Umans'kiĭ, E. S.; Kvitka, O. L.; and Agar'ov, V. A. Method of initial functions in the axisymmetric problem of the theory of elasticity. Dopovidi Akad. Nauk Ukrain. RSR **1958**, 1167–1171. (Ukrainian. Russian and English summaries)

A general solution to the axisymmetric isotropic thermoelastic problem, in terms of four functions, following a method by Vlasov is given. It leads to ordinary differential equations and can be extended to the torsion of solids of revolution.

J. Nowinski (Madison, Wis.)

7418:

Bloh, V. I. Expression of the general solution of the static problem of the theory of elasticity for an isotropic body by applying plane harmonic functions. Dopovidi Akad. Nauk Ukrain. RSR **1958**, 1172–1176. (Ukrainian. Russian and English summaries)

It can be shown that a spatial biharmonic function B may be represented by spatial harmonic functions F, $\mathbf{G}$, H and Π in a fairly general form

$$(1)\qquad B=F+\mathbf{r}\cdot\mathbf{G}+r^2H+\mathbf{rrr}\cdots\Pi,$$

where $\mathbf{r}$ is a positional vector and Π a tensor of rank three. If only plane harmonic functions are used, B may be made dependent on six functions of a complex variable and their conjugates (form 2). The author represents the elastic displacement vector $\mathbf{u}$ in the form (3) $\mathbf{u}=\mathbf{v}-\nabla B$, where $\mathbf{v}$ is a harmonic vector and $\nabla=2(\mathbf{e}(\partial/\partial\zeta)+\bar{\mathbf{e}}(\partial/\partial\bar{\zeta}))+\mathbf{k}(\partial/\partial z)$, $\mathbf{e}$ and $\bar{\mathbf{e}}$ being the conjugate complex coordinate vectors. If specified for the plane problem, form (3) reduces to the Kolosov-Muskhelishvili form.

J. Nowinski (Madison, Wis.)

7419:

Amenzade, Iu. A. Local torsional stresses in a prismatic circular beam with an eccentric elliptical hole. Soviet Physics. Dokl. **119** (3) (1958), 446–450 (1118–1121 Dokl. Akad. Nauk SSSR).

The problem is solved using complex function theory. Numerical values are given for the torsional rigidity and for some stresses. *A. E. Green* (Newcastle-upon-Tyne)

7420:

Boyce, William; and Handelman, George. Vibrations of twisted beams. II. Quart. Appl. Math. **16** (1958), 385–395.

[For part I see R. C. Di Prima and G. H. Handelman, same Quart. **12** (1954), 241–259; MR **16**, 311.] An assortment of qualitative properties of the energy form for the transverse vibrations of a twisted beam rotating about an axis is discussed.

C. R. DePrima (Pasadena, Calif.)

7421:

★**Lee, C. W.; and Donnell, L. H. A study of thick plates under tangential loads applied on the faces.** Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 401–409. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

"The three-dimensional stresses and displacements in a thick plate under any tangential loads applied on both surfaces are obtained in the form of infinite series, whose first terms represent the classical thin plate theory for antisymmetric loading, and the plane-stress solution for symmetric loading. The general terms for these series solutions and also for the previously studied case of normal loads are included." (Authors' summary)

A. P. Coppa (Philadelphia, Pa.)

7422:

Iškova, A. G. Bending of a circular plate and an infinite strip lying on an elastic half-plane. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1958**, no. 10, 87–91. (Russian)

The solutions of such elastic problems as the bending of a circular plate or a semi-infinite strip, for prescribed loads, are expansional and reduce to infinite systems of the form:

$$\sum_{n=0}^{\infty} \frac{\alpha_{\mu n}}{2n-2m-1} = \beta_{\mu m} \ (\mu=1, 2, \cdots; m=0, 1, \cdots).$$

The solution, previously given empirically [Iškova, Moskov. Gos. Univ. Uč. Zap. **152**, Meh. 3 (1951), 202–225; MR **14**, 701],

$$\alpha_{\mu n} = \frac{4}{\pi} \frac{n!}{(n-\frac{1}{2})!} \sum_{s=0}^{\infty} \frac{(s+\frac{1}{2})!}{s!} \frac{\beta_{\mu s}}{2n-2s-1}$$

is here established. *H. G. Baerwald* (Albuquerque, N.M.)

7423:

Tamate, O. Einfluss einer unendlichen Reihe gleicher Kreislöcher auf die Durchbiegung einer dünnen Platte. Z. Angew. Math. Mech. **37** (1957), 431–441. (English, French and Russian summaries)

This paper is concerned with the influence of an infinite series of identical circular holes on the bending of an infinite plate within the scope of the classical (Poisson-Kirchhoff) theory of thin plates, and with the use of the method of complex representation due to Muskhelishvili. The solution is formally given as an infinite series, and, specifically, the following cases are investigated: (a) cylindrical bending about the x-axis; (b) cylindrical bending about the y-axis; (c) plain bending about the x-axis; and (d) plain bending about the y-axis. By successive approximations numerical results are obtained for each of the above cases for several values of λ (ratio of hole radius to distance between centers of the holes). The author observes that the convergence of the solution may be established at least for $\lambda<0.35$ in a manner similar to that employed by Howland [Proc. Roy. Soc. London Ser. A **148** (1935), 471–491]. The reduction of the solution to the work of Goodier on the bending of an infinite plate with a single circular hole [Philos. Mag. (7) **22** (1936), 68–80] is also shown. *P. M. Naghdi* (Berkeley, Calif.)

7424:

Oravas, Gunhard-Aestius. On the theory of nearly spherical thin shells. Z. Angew. Math. Mech. **38** (1958), 379–386. (German, French and Russian summaries)

7425:

★**Yu, Yi-Yuan. On the Donnell equations and Donnell-type equations of thin cylindrical shells.** Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 479–487. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

Let (x, s, ζ) be the triply orthogonal coordinate system with ζ directed along the positive normal to the middle surface of the circular cylindrical shell, and x and s measured, respectively, along the generator and in the circumferential direction. Further, let the components of the displacement vector be taken as

(1) $\quad U_x=u_x+\zeta\beta_x,\ U_s=u_s+\zeta\beta_s,\ U_\zeta=w,$

where u_x, β_x, w, etc., are all functions of x, s, and time t. With approximations similar to those of the theory of shallow shells, the differential equations of motion of cylindrical shells, with the effects of both transverse shear deformation and rotatory inertia included, may be put in the form

(2a) $\quad L_1 w=0,$

(2b) $\quad L_2\begin{Bmatrix} u_x \\ u_s \end{Bmatrix} = \begin{Bmatrix} f_1(w) \\ f_2(w) \end{Bmatrix},$

(2c) $\quad L_3\begin{Bmatrix} \beta_x \\ \beta_s \end{Bmatrix} = \begin{Bmatrix} \partial/\partial x \\ \partial/\partial s \end{Bmatrix} g(u_x, u_s, w),$

where the operators L_1, L_2, and L_3 (containing partial derivatives with respect to time, as well as the space variables x and s) are of the eighth, fourth and second orders, respectively, in the space variables. When the effects of transverse shear deformation and rotatory inertia are neglected in (2), the resulting differential equations correspond to those given by Donnell. In connection with a study of propagation of elastic waves in cylindrical shells, by the reviewer and R. M. Cooper, the system of equations (2) was included in J. Acoust. Soc. Amer. **28** (1956), 56–63, where it is also remarked that the homogeneous differential equations associated with (2b) do not yield independent solutions, while the homogeneous solutions for β_x and β_s deducible from (2c) satisfy the same single second order partial differential equations; thus the independent order of the system of equations (2) is ten, while that of Donnell is eight.

In the paper under review, following some discussions of Donnell equations and "Donnell-type equations", utilizing the basic equations which led to (2) and by further manipulations, the author obtains a system of equations for circular cylindrical shells equivalent to (2). {Reviewer's Note: the first of equations (12) of the paper is of tenth order, while the remaining equations (12) of the paper do not yield independent solutions; the advantage, if any, of the derived equations over those of system (2) is not apparent. The paper contains a number of misleading and incorrect statements; to cite an example, with reference to the set of equations (2) it is stated "In this set, however, an operator in the first equations that involves only w is missing. As a result, the equation is only of the eighth order with respect to both the time and space variables, instead of the tenth, which is the correct order. If this equation were used in the investigation of the general vibration of cylindrical shells, only four instead of five frequencies would be obtained, and in general all the five conditions at a shell boundary could not be satisfied." That this is not the case and that, for the asymmetric problem of wave propagation, equations (2) give rise to five branches of the dispersion curve can be

seen in a more recent paper by R. M. Cooper and the reviewer [J. Acoust. Soc. Amer. **29** (1957), 1365–1373; MR **19**, 1000]. References on the boundary conditions for shells with the effect of transverse shear deformation included may be found in an article by the reviewer [Appl. Mech. Rev. **9** (1956), 365–368]; see also a paper by R. M. Cooper [J. Appl. Mech. **24** (1957), 553–558; MR **19**, 904], where the correct boundary conditions for shallow shells are stated.} *P. M. Naghdi* (Berkeley, Calif.)

7426:

★Chang, C. S. Energy dissipation in longitudinal vibration. Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 109–116. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

This paper deals with the longitudinal vibrations of a finite bar with a viscous damper at one end and arbitrary exciting forces at the other. Arbitrary initial displacements and velocities are also considered. The problem is first split into steady-state and transient parts, the latter having time-independent end conditions. The transient solution is then found by more or less classical methods appropriate to the one dimensional wave equation. The steady-state solution is determined by a Fourier type analysis when the exciting force is periodic. When this is not the case, a series expansion is given, the terms of which are essentially values of the exciting force at earlier times. *W. E. Boyce* (Troy, N.Y.)

7427:

Hu, Hai-chang. On two variational principles about the natural frequencies of elastic bodies. Sci. Sinica **7** (1958), 298–312.

This largely expository paper is a discussion of two minimum principles in elastic vibrations. One makes use of a class of statically admissible comparison functions which satisfy the equations of motion and stress boundary conditions. The compatibility equations and displacement boundary conditions are satisfied by the solution of the minimum problem. The other involves no assumptions (except sufficient continuity) on the comparison functions, and all of the relative equations are satisfied by the minimizing set of functions. It is also shown that these general principles reduce to the well-known Rayleigh quotient for various problems involving beams and plates. *W. E. Boyce* (Troy, N.Y.)

7428:

Ivovich, V. A. Subharmonic oscillations of rods with nonlinear inertia. Soviet Physics. Dokl. **119** (3) (1958), 434–437 (237–240 Dokl. Akad. Nauk SSSR).

The paper considers the subharmonic solutions of the non-linear equation which describes the transverse vibrations of an elastic rod with hinge-supported ends when the movable support carries a concentrated mass M. If m is the mass per unit length of the rod, l is its length, a pulsating force $P\cos pt$ is applied at the midpoint of the rod and the elastic curve is given by $y=q(t)\sin \pi x/l$, the nonlinear equation is

$$\frac{d^2q}{dt^2}+2\varepsilon\frac{dq}{dt}+\kappa q\frac{d^2}{dt^2}(q^2)+\omega^2q=\frac{2P}{ml}\cos pt,$$

where ε=coefficient of viscous damping and $\kappa=\pi^4M/4ml^3$. For this equation subharmonic resonance of orders $\frac{1}{2}$ and $\frac{1}{3}$ is investigated by considering the first few terms in the Fourier series for the solution. *G. B. Warburton* (Edinburgh)

7429:

Carmichael, T. E. The vibration of a rectangular plate with edges elastically restrained against rotation. Quart. J. Mech. Appl. Math. **12** (1959), 29–42.

The Rayleigh-Ritz method is used to analyse the vibration of a rectangular plate with edges elastically restrained against rotation. (From the author's summary.) *G. B. Warburton* (Edinburgh)

7430:

Terazawa, Kazuo; and Matsuura, Yoshikazu. Transverse vibration of higher frequencies of beams of uniform cross section taking into account the effect of shear. Tech. Rep. Osaka Univ. **8** (1958), 281–297.

Solutions of Timoshenko's equation of flexural vibrations of beams are studied in a range of frequencies high enough for the shear term to be important but low enough for the rotatory inertia term to be negligible. Both effects have been examined in great detail in the theory of plates in which the equations reduce to the same form as Timoshenko's in the case of plane strain or stress [e.g., R. D. Mindlin, J. Appl. Phys. **22** (1951), 316–323; MR **12**, 771; J. Appl. Mech. **18** (1951), 31–38; and subsequent papers in the same journals]. *R. D. Mindlin* (New York, N.Y.)

7431:

Tolokonnikov, L. A. Critical pressures on a circular plate. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1958**, no. 10, 77–86. (Russian)

Axisymmetric states of equilibrium of a circular disk (of thickness h and radius a) acted upon, on its cylindrical surface, by an external pressure q are considered. The disk is clamped at the circumference, except for radial displacement which is unhampered. As is known, three alternative phenomena may occur: (a) an elastic buckling induced by $q_{\mathrm{el.crit.}}$ for sufficiently small ratios of h/a, (b) an elastic-plastic buckling induced by $q_{\mathrm{el.pl.crit.}}$ for greater ratios of h/a, and (c) no loss of stability up to the failure of the disk, starting with some minimum limit value of h/a.

Using finite deformation theory, the Lagrange variational equation is derived for stability of the basic state of equilibrium. In this state a plane state of stress in the disk is assumed, the stress being represented by the invariants of stress, and the deformations by the symmetric invariants: the relative change of volume, and the algebraic invariants $\mathfrak{d}_t$ and β (the phase) of the deviator of logarithmic extensions of the change of form. Assuming the existence of the stress potential and $\beta=\varphi$, where φ is the angle of the octahedral shearing stress τ_t, one obtains the relation $\tau_t=\tau_t(\mathfrak{d}_t)$, (taken to be linear for $\mathfrak{d}_t\leq\mathfrak{d}_{\mathrm{yield}}$, and in direct proportion to $\mathfrak{d}_{\mathrm{yield}}^{1-\kappa}\mathfrak{d}_t^{\kappa}$ (κ=constant) for $\mathfrak{d}_t>\mathfrak{d}_{\mathrm{yield}}$), and the corresponding equations of state. It is assumed that after the loss of stability, in the elastic range, the fibers in the principal directions remain orthogonal, and two zones, (a) of active deformation ($\delta\mathfrak{d}_t^0\geqq 0$), and (b) of unloading ($\delta\mathfrak{d}_t^0\leqq 0$), can be differentiated (nought denotes the basic state). For both zones different equations for (radial, hoop and a third normal) stress are derived. A nonlinear relation (NE) between $(h/a)^2$ and the buckling pressure q/G (G=modulus of rigidity) obtained shows clearly the existence of a limit value of h/a beyond which the disk is elastically stable. A straight line representing the usual equation for the buckling pressure is tangent to the curve (NE), at the origin of coordinates.

Also a critical curve (NEP) for the elastic-plastic buckling of the disk is found, the respective phenomenon

requiring, however, the existence of a pronounced region of unloading. The formulae derived for NE and NEP show a discontinuity at the yield point.

J. Nowinski (Madison, Wis.)

7432:

★**Masur, E. F. On the analysis of buckled plates.** Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 411–417. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

An analysis of the postbuckling behaviour of plates is presented, based on the large deflection theory of von Kármán. The middle surface (membrane) stresses are shown to obey a minimum energy principle and this, together with the idea of a stress-function space, is used to develop an iterative procedure for the determination of approximate solutions. An error estimate is provided at each stage through the establishment of upper and lower bounds. The procedure is illustrated by the example of a simply supported circular plate which buckles into a mode of circular symmetry under the action of a uniform radial pressure.

J. E. Adkins (Nottingham)

7433:

Petrašen', G. I. The investigations of the propagation of elastic waves. Vestnik Leningrad. Univ. **13** (1958), no. 22, 119–136. (Russian. English summary)

This is a review of recent Soviet accomplishments in the theory of wave propagation in elastic solids, with particular reference to transient sources. These advances have been based upon a systematic analytical and numerical exploitation of various approaches, such as those of first- and higher-order ray optics, and contour integration methods combined with the use of special functions (inhomogeneous media). Of course, these methods usually lead to very heavy algebraic manipulations, as is to be expected from the nature of the problem. But it appears that the results can be usefully displayed in approximate form, as "engineering-type" formulae (the writer's expression). It is stated that application of these results to practical problems in seismic exploration has already been successful, and will soon be utilized in a routine fashion for analyzing the amplitude information contained in seismic records.

I. Tolstoy (Dobbs Ferry, N.Y.)

7434:

Cepelev, N. V. Reflection of elastic waves in a nonhomogeneous medium. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1959**, 11–17. (Russian)

An approximate method of obtaining reflection coefficients at discontinuities of the velocity gradient and of its derivatives in solid elastic media.

I. Tolstoy (Dobbs Ferry, N.Y.)

7435:

Čekin, B. S. Reflection and refraction of seismic waves on a weak boundary of separation. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1959**, 18–26. (Russian)

This paper examines the same problem as Cepelev's article [preceding abstract], with some additional emphasis on higher order terms in the expansion of the field in inverse powers of the frequency.

I. Tolstoy (Dobbs Ferry, N.Y.)

7436:

Fischer, H. C. On longitudinal impact. I. Fundamental cases of one-dimensional elastic impact. Theories and experiments. Appl. Sci. Res. A 8 (1959), 105–139.

This paper gives a review of the theory of longitudinal elastic impact on slender bars and describes some experimental work in this field. The effect of the lateral inertia of the bars is not discussed and plane strain conditions are assumed throughout. Solution by means of the method of characteristics is considered in some detail.

H. Kolsky (Fort Halstead)

7437:

Manacorda, Tristano. Sul comportamento meccanico di una classe di corpi naturali. Riv. Mat. Univ. Parma 8 (1957), 15–25.

The author sets up heuristic considerations leading to a visco-elastic theory of the Meyer-Voigt type. The elastic stress is taken as a general isotropic function of the finite strain; the viscous stress is a quasi-linear function of the rate of deformation; and the two are superposed. The author shows that the characteristic surface elements of the system resulting from linearizing his equations cannot be real.

C. Truesdell (Bloomington, Ind.)

7438:

Slibar, Alfred; and Paslay, Paul R. Retarded flow of Bingham materials. J. Appl. Mech. **26** (1959), 107–113.

Solutions are given for the following three cases of the retarded flow of rigid-viscous (Bingham) materials: (a) laminar flow between parallel plates; (b) Couette flow between concentric circular cylinders; (c) axial (Poiseuille) flow through a circular pipe. In the latter case the central core of the material moves as a rigid material. Two different unloading situations are investigated for case (b).

J. E. Adkins (Nottingham)

7439:

Shermergor, T. D. Thermodynamic theory of relaxation processes. Soviet Physics. Tech. Phys. **28** (3) (1958), 606–613 (647–654 Ž. Tehn. Fiz.).

The stress-strain relations for an isotropic viscoelastic solid are here derived by the methods of irreversible thermodynamics. The stress tensor is expressed as the gradient of the free energy and the strain tensor as the gradient of the thermodynamic potential. It is then shown that the relation between stress and strain for a general deformation can be expressed in integral form with two 'memory' functions which apply to shear and dilatation respectively. These can be expressed either in terms of creep or of stress relaxation at constant strain. For isothermal deformation the treatment leads to Boltzmann's superposition principle. The stress-strain relation can also be expressed in derivative form but this becomes cumbersome if more than one or two relaxation times are operative.

H. Kolsky (Fort Halstead)

7440:

Petrova, S. G. On the boundary value problems of nonlinear elasticity. Vestnik Leningrad. Univ. **14** (1959), no. 1, 57–78. (Russian. English summary)

Let $E=(\varepsilon_{ik})$ and $T=(\tau_{ik})$ denote the strain and stress tensors, respectively, and let $3\varepsilon=\sum \varepsilon_{ii}$. For elastic-plastic material under small deformations (shears, elongations and rotations are small) the stress-strain relation is approximately $T=\psi(\Gamma^2)E+(k-\frac{1}{3}\psi(\Gamma^2))\varepsilon I$, where I is the unit matrix, $k=\lambda+\frac{2}{3}\mu$; $\Gamma^2=\frac{2}{3}\sum_{i<j}(\varepsilon_{ii}-\varepsilon_{jj})^2+\frac{4}{3}\sum_{i<j}(\varepsilon_{ij})^2$, $0\leqq\psi(\Gamma^2)\leqq 2\mu$ and $\psi\to 2\mu$, $\psi'\to 0$ as $\Gamma^2\to 0$. Let K denote a given force function. The first boundary problem for the displacement vector u is (1) $(\operatorname{div} T+K)u=0$ in a bounded domain and $u=0$ on its boundary. Writing $\psi(\Gamma^2)=2\mu(1-\kappa w(\Gamma^2))$ the author proves existence of a unique solution of (1) provided κ is small. The proof is based

upon known existence theory for linear elasticity or, more generally, for linear strongly elliptic systems and upon results of L. V. Kantorovich [Uspehi Mat. Nauk SSSR (N.S.) 3 (1948), no. 6 (28), 89–185; MR 10, 380]. The last paper also suggests an approximation method for (1). For the case of two dimensions the author gives a second proof. He first considers the special case of a circular domain and makes use of fundamental solutions. The general case is then derived by conformal mappings.
A. Friedman (Berkeley, Calif.)

7441:

★Perrone, Nicholas; and Hodge, P. G., Jr. Strain hardening solutions with generalized kinematic models. Proceedings of the Third U. S. National Congress of Applied Mechanics, Brown University, Providence, R. I., June 11–14, 1958, pp. 641–648. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

A rigid-plastic *I*-beam with unequal flanges is simply supported and centrally loaded. The deformation is assumed so small that dimensional changes can be disregarded, and the beam is so long that the stress is essentially uniaxial. Linear workhardening in uniaxial straining is postulated, with unequal tensile and compressive yield stresses. After obtaining the solution straightforwardly, the authors consider an alternative method. This is based on the yield relation between bending moment and axial force (the latter is zero in the actual problem), linked via a plastic potential hypothesis with the two corresponding generalized strains, together with a workhardening law in two independent variables. There is considerable arbitrariness in the choice of the latter, and the authors use this special problem to suggest certain requirements in a general theory of biaxial hardening.
R. Hill (Nottingham)

7442:

★Murch, S. A.; and Naghdi, P. M. On the infinite elastic, perfectly plastic wedge under uniform surface tractions. Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 611–624. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

An infinite elastic perfectly plastic wedge satisfying Tresca's yield criterion, is subject to uniform normal and shearing forces on both faces. The authors classify the angles of onset of yield in plane strain for different combinations of load, and different wedge angles; for greater loads they find that there are two possible forms of the solution in the plastic regions. One complete plane strain and one complete plane stress solution are given.
D. R. Bland (Manchester)

7443:

Olszak, Wacław; and Perzyna, Piotr. Criteria of validity of variational theorems in mechanics of inelastic non-homogeneous anisotropic deformable bodies. Arch. Mech. Stos. 10 (1958), 559–563. (Polish and Russian summaries)

Using results from the calculus of variations, the authors extend the validity of a known variational theorem of plasticity so as to include a theory of considerable generality but limited, apparently, to small deformations. Let T^{ijkl} and $\mathring{T}_{ijkl}$ be functions of position, time, stress (τ^{pq}), and strain (ε_{rs}) such that the forms $2\Pi=T^{ijkl}\dot{\varepsilon}_{ij}\dot{\varepsilon}_{kl}$ and $2\Omega=\mathring{T}_{ijkl}\dot{\tau}^{ij}\dot{\tau}^{kl}$ are positive definite and such that $T^{ijkl}\mathring{T}_{prst}=\delta_p{}^i\delta_r{}^j\delta_s{}^k\delta_t{}^l$. Dots denote, apparently, the local time derivative, and

$$\dot{\tau}^{ij}=\frac{\partial\Pi}{\partial\dot{\varepsilon}_{ij}},\quad \dot{\varepsilon}_{ij}=\frac{\partial\Omega}{\partial\dot{\tau}^{ij}},$$

so that there is a full analogy to the canonical formalism. The authors show that several of the previously derived variational theorems for plasticity and visco-elasticity are included in their result.
C. Truesdell (Bloomington, Ind.)

7444:

Ržanicyn, A. R. Plastic deformations of a tube with axisymmetric loading. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk 1958, no. 9, 60–65. (Russian)

Plastic deformations of a circular cylindrical shell composed of two layers of equal thickness and at constant distance apart, acted upon by an axisymmetric load, are considered assuming Huber-Mises-Hencky material. It is shown that the plastic state of the tube may be represented by two second order ordinary differential equations for the longitudinal bending moment m (per unit length of the middle line of a cross-section) and for the radial displacement. The relation between m and the hoop stress is represented by a circle. *J. Nowinski* (Madison, Wis.)

7445:

Schlechtweg, H. Zur Identität von Gleitlinien und Charakteristiken. Z. Angew. Math. Mech. 39 (1959), 82.

7446:

Matschinski, Matthias. De la plasticité "linéaire". C. R. Acad. Sci. Paris 248 (1959), 636–639.

The author quotes many fields of mathematical physics where the basic equations can always be linearized provided certain quantities are sufficiently small. He states that this must always be true "except in the case where a hypothesis of discontinuity is included". In particular, he asserts that it must be possible to linearize the equations of plasticity. Assuming that the problem is linear, tensorially invariant, and can contain only certain derivatives, the author then claims that the density must satisfy an equation of the form

$$\theta'+\varepsilon\Delta\theta'+\operatorname{grad}\xi_2\cdot\operatorname{grad}\theta'=h\Delta\theta+\operatorname{grad}\xi_1\cdot\operatorname{grad}\theta+F$$

where θ is the density, primes denote time derivatives, Δ is the Laplacian, F the force, ξ_1 and ξ_2 "tensorial constants of the material," and ε and h are not defined.
P. G. Hodge, Jr. (Chicago, Ill.)

7447:

Storchi, Edoardo. Sforzi plastici in una membrana piana di spessore variabile. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 24 (1958), 685–692.

The author considers a rigid-plastic plate in plane stress and treats separately the cases of non-thickening and of thickening. In the reviewer's opinion the author makes simplifying assumptions in both cases that reduce the value of the results. In the first case $\sigma_{zz}=0$ and $d\varepsilon_{zz}{}^p=0$ imply $\sigma_{xx}+\sigma_{yy}=0$; in the second case $\sigma_{xz}=\sigma_{yz}=\sigma_{zz}=0$ implies $\sigma_{xx}+\sigma_{yy}=Ax+By+C$ (the Mises criterion is used, x, y is the plane of plane stress and A, B and C are constants). In general, the boundary conditions of the problem will not satisfy either of these conditions. The restrictive condition in the second case also occurs in plane stress in elasticity, where it can be avoided by the use of generalised plane stress. It would therefore be interesting to re-examine the author's problem in generalised plane stress instead of in plane stress.
D. R. Bland (Manchester)

7448:

Drucker, D. C.; and Shield, R. T. Limit analysis of symmetrically loaded thin shells of revolution. J. Appl. Mech. 26 (1959), 61–68.

The stress state of a symmetrically loaded thin shell of revolution is specified by two direct stresses N_θ and N_ϕ, and two bending moments M_θ and M_ϕ. These must satisfy two equations of equilibrium and, for a perfectly plastic material, a yield condition. By an order of magnitude argument, the authors show that M_θ and M_ϕ contribute relatively little to the equilibrium equations, compared to N_θ, N_ϕ, and the derivative $dM_\phi/d\phi$, provided that the shell thickness is small compared to the distance from the axis. They suggest approximating these equations by ignoring M_θ and either ignoring or including M_ϕ depending upon the particular problem. Since M_θ can always be eliminated from the yield condition, the result is a problem in three stress variables rather than four. The yield condition is now the same as that of the circular cylindrical shell; approximations suitable for the cylinder will also be useful for the shell of revolution.

The theory described above is used to find approximations to the load-carrying capacity of a pressure vessel consisting of a cylinder with a spherical head connected by a toroidal knuckle. *P. G. Hodge, Jr.* (Chicago, Ill.)

7449:

★**Hoff, N. J. A survey of the theories of creep buckling.** Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 29–49. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

A survey is presented of the theories of buckling of structural elements whose material is subject to creep deformation. Two fundamentally different approaches to the solution of the buckling problem are discussed. In one the structural element is assumed to be perfect and perfectly centered under the loads and buckling is initiated by a disturbance in equilibrium configuration. In the second the creep deformations begin in consequence of the deviation of the unloaded centerline or median plane of the structural element from the line of load application. In both cases the element has a finite lifetime. A critical time of creep buckling is defined beyond which the element cannot be used to carry the prescribed loads.

The paper gives a valuable discussion of available theories; the bibliography with 50 entries is of particular value. *W. T. Koiter* (Delft)

7450:

★**Litwiniszyn, J. Fundamental principles of the mechanics of stochastic media.** Proceedings of the Third Congress on Theoretical and Applied Mechanics, Bangalore, December 24–27, 1957, pp. 93–110. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur, 1958. xi+362 pp.

The paper presents essentially the author's work reported in Arch. Mech. Stos. **8** (1956), 393–411 [MR **19**, 599], with some examples, comparisons with experimental work, and more rigor in the derivation of the equations.

W. Freiberger (Providence, R.I.)

7451:

★**Wilhoit, J. C., Jr. Elastic-plastic stresses in rings under steady state radial temperature variation.** Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 693–700. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

Elastic-plastic states in a ring with insulated upper and lower faces under axially symmetric steady state temperature field are considered using Tresca yield condition. Elastic moduli, coefficient of thermal expansion and yield stress are assumed to be temperature insensitive. It is found that with increasing temperature difference a second plastic region begins to form at the outer surface and moves inward. The solution is obtained by simple mathematical treatment. *J. Nowinski* (Madison, Wis.)

7452:

★**Vinson, Jack R. Thermal stresses in laminated circular plates.** Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 467–471. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

An elastic circular plate composed of two laminae of different material and of different but constant thickness, bonded at the common face, is investigated, the temperature field being arbitrary but axially symmetric. From sixteen equations for the same number of unknown quantities, i.e., for five stress resultants and couples, deflection and radial displacement u_0 of the middle plane (all for each lamina), as well as for the shear and normal stress at the joint, two equations are derived: (a) one concerning normal stress at the joint, and the other (b) a nonhomogeneous biharmonic equation for the (common) deflection. Expanding the temperature function into the Fourier series permits an effective integration of (b), the constants of integration being determined from the boundary conditions. For known deflection, u_0 and consequently the stress resultants and couples can be found. Small deflection theory of thin plates is used throughout.

J. Nowinski (Madison, Wis.)

7453:

★**Langhaar, H. L.; and Boresi, A. P. Strain energy and equilibrium of a shell subjected to arbitrary temperature distribution.** Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 393–399. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

A general small-deflection theory of thermal stresses in elastic shells is developed assuming that normals to the surface remain straight and normal. Curvilinear coordinates (x, y, z) are used, where (x, y) are the lines of principal curvature and z is normal to the middle surface. The tractions, bending moments and strain energy are expressed in terms of the displacements and the temperature $T(x, y)$. *E. H. Mansfield* (Farnborough)

7454:

Ignaczak, Józef. The stresses due to a nucleus of thermoelastic strain in a semi-infinite plate containing a semicircular notch. Arch. Mech. Stos. **10** (1958), 707–713. (Polish and Russian summaries)

Consider a semi-infinite elastic plate, the straight edge of which has a semi-circular notch. The paper aims at the

two-dimensional solution appropriate to a center of dilatation which lies on the axis of symmetry in the interior of the plate. Starting with the known elementary solution for the center of dilatation inside a half-plane, the author reduces the residual problem to the solution of an infinite system of linear algebraic equations. This is accomplished by means of a scheme originally employed by Maunsell [Philos. Mag., Ser. 7, **21** (1936), 765–773] in connection with the problem of the notched plate under tension. {A solution in integral form to the present residual problem could be obtained on the basis of C. B. Ling's [J. Math. Phys. **26** (1947), 284–289; MR **9**, 481] approach to Maunsell's problem.}

E. Sternberg (Providence, R.I.)

7455:

Sneddon, I. N. The propagation of thermal stresses in thin metallic rods. Proc. Roy. Soc. Edinburgh Sect. A **65** (1959), 121–142.

The system of partial differential equations $\sigma_x = au_{tt}$, $\sigma = u_x - b\theta$, $\theta_{xx} = f\theta_t + gu_{xt}$ is developed for the longitudinal displacements $u(x, t)$ and stress $\sigma(x, t)$, and temperatures $\theta(x, t)$ in thin metal rods with thermally insulated lateral surfaces. The equations here are written in dimensionless units and the coefficients a, b, f and g are constants that depend on the elastic and thermal properties of the material. Modifications of the system are given in case linear heat transfer takes place at the lateral surface of the rod or in case longitudinal body forces are present.

Steady sinusoidal variations of u, σ and θ with respect to time t in a rod of infinite length are considered first. Phase velocity and attenuation of thermal and elastic waves are found to depend sharply on a certain critical frequency. The attenuation and phase velocity are computed and tabulated for bars of four different metals: aluminum, copper, iron and lead, for a range of frequencies.

For semi-infinite bars ($x \geq 0$), end conditions that prescribe sinusoidal variations at $x=0$ of either the displacement, the stress, the temperature or the flux of heat are considered. Formulas are obtained for $u(x, t)$, $\sigma(x, t)$ and $\theta(x, t)$. For more general end conditions, Laplace transforms are used. In particular, when the stress at the end is a step function of t and the entire surface of the bar is insulated and the initial temperature is uniform, a formula is found for the temperature variation at the end, with tabulated numerical data for bars of the four metals named above. The paper ends with a consideration of bars of finite length and a discussion of approximate methods of solving the system of equations.

R. V. Churchill (Ann Arbor, Mich.)

STRUCTURE OF MATTER

See also 7405, 7560.

7456:

Avak'yanc, G. M. Theory of the behavior of semiconductors in strong electric fields. Izv. Akad. Nauk UzSSR. Ser. Fiz.-Mat. **1958**, no. 4, 23–36. (Russian. Uzbek summary)

7457:

Brout, R. Sum rule for lattice vibrations in ionic crystals. Phys. Rev. (2) **113** (1959), 43–44.

Les carrés des pulsations des oscillations harmoniques faites par les atomes sont égaux aux valeurs caractéristiques d'une matrice hermitienne, G, fonction uniquement du vecteur d'onde; et la somme de ces carrés est donc égale à la trace de cette matrice. S'appuyant sur les propriétés des fonctions harmoniques et ne retenant que les forces répulsives entre les atomes au contact, l'auteur démontre que pour un cristal ionique du type sel gemme (NaCl) la trace de la matrice G est constante (indépendante du vecteur d'onde), entièrement déterminée par le coefficient de compressibilité, par les masses individuelles des ions et par la distance des ions immédiatement voisins. De même, en faisant des approximations, il trouve que pour un cristal métallique, cubique centré, la trace de la matrice G est aussi constante, égale au carré de la pulsation classique des oscillations du plasma électronique.

J. Laval (Paris)

7458:

Vonsovsky, S. V.; and Turov, E. A. Some problems of phenomenological theory of Ferro- and antiferromagnetism. J. Appl. Phys. **30** (1959), no. 4, Supplement, 9S–18S.

A general review is given of a phenomenological approach to the theory of ferro- and antiferromagnetism. The principal mathematical tool involved is the use of arguments based on crystalline symmetry properties to provide information on the Hamiltonian operator.

E. L. Hill (Minneapolis, Minn.)

FLUID MECHANICS, ACOUSTICS

See also 7438, 7488, 7493, 7499, 7511, 7514, 7588.

7459:

Müller, Henning. Zur Frage der Charakterisierung stationärer Bewegungen in der Hydrodynamik. Z. Angew. Math. Phys. **9a** (1958), 389–392. (English summary)

First the author follows through the steps leading to the Helmholtz-Korteweg theorem of minimum energy dissipation, retaining the inertial terms. Thus he derives the interesting inequality

$$\int \Phi(v') - 2\rho \int (v_k - v_k')v_l v_{k,l} > \int \Phi(v)$$

over a region in which v is a velocity field satisfying the Navier-Stokes equations for steady flow of an incompressible fluid, Φ is the dissipation function, and v' is another velocity field which equals v upon the boundary. The Helmholtz-Korteweg theorem corresponds to neglect of the second integral on the left. {Much fuller results of this kind appear in the article by J. Serrin, [Handbuch der Physik, Bd. 8/1, pp. 125–263, Springer, Berlin–Göttingen–Heidelberg, 1959]. The restriction to steady flow is unnecessary, a simple alteration extends the result to compressible fluids, etc.} The author works out an application claimed to correspond to the motion of a "simple macromolecule in an inhomogeneous field of flow".

C. Truesdell (Bloomington, Ind.)

7460:

Dumitrescu, D.; et Ionescu, Dan Gh. Méthodes numériques pour l'étude des mouvements à symétrie axiale des fluides parfaits. Acad. R. P. Romîne. Stud. Cerc. Mec. Apl. **9** (1958), 919–935. (Romanian. Russian and French summaries)

7461:

Mitra, M. K. Resistance on a sphere due to a circular vortex filament in an uniform flow of a perfect liquid. Z. Angew. Math. Mech. **39** (1959), 25–30. (German, French and Russian summaries)

The author determines the resistance on a sphere due to a circular vortex filament in a uniform flow of a perfect fluid. The solution is given by a series containing Legendre's polynomials. *M. Schechter* (New York, N.Y.)

7462:

Viguier, G. Les équations du mouvement des fluides visqueux dans le cas de gradients de vitesses élevés. Bul. Inst. Politech. Iaşi **4** (1949), 203–221. (1 insert)

The author investigates a number of special viscous flows by means of the equations of M. Girault, derived on the hypothesis that the viscous stresses contain not only terms of the first degree in the deformation tensor, but also those of the third degree. From the numerical results shown, it appears that the effects on the third order terms are considerable and, if true, should be observed. *Y. H. Kuo* (Peking)

7463:

Stewartson, K. On Goldstein's theory of laminar separation. Quart. J. Mech. Appl. Math. **11** (1958), 399–410.

Goldstein [Quart. J. Mech. Appl. Math. **1** (1948), 43–69; MR **10**, 270] has an expansion of the stream function ψ in the form $\xi^3\sum_0^\infty \xi^r f_r(\eta)$, with $\xi=x^{1/4}$, $\eta\propto y/\xi$, where x is the dimensionless upstream distance from the separation point. His basic assumption is that $u(y)$, the velocity profile at $x=0$, is analytic; but $f_6(\eta)$ et seq. can only be determined if a certain integral relation involving the earlier f's holds, the validity of which is not beyond doubt. However, if it holds, then, remarkably, all boundary layers with the same dimensionless pressure distribution $1+\sum_1^\infty P_r x^r$ would have the same asymptotic $\psi(\xi,\eta)$.

In the present paper the author derives by methods explained in J. Math. Phys. **36** (1957), 179–191 [MR **19**, 1219] the following results. 1. The leading terms of ψ are $\xi^3\sum_0^5 \xi^r f_r(\eta)+\xi^8\log\xi[F_5(\eta)+\xi F_6(\eta)]$, and the actual ψ is a double power series in ξ and $\log\xi$ with coefficients functions of η. 2. Unless $f_1''(0)=0$, an infinity of constants, namely $f_r''(0)$ for $r=4n+1$, cannot be determined from the pressure distribution but, presumably, must be found from a given upstream profile $u(x_0,y)$, $x_0>0$; all this is true whether or not the above-mentioned integral relation holds. 3. The form of ψ precludes the existence of the power-series expansion of $u(y)$ (assumed by Goldstein); nevertheless $x=0$ need not be a singular line. 4. The downstream continuation of ψ seems only feasible if $f_1''(0)=0$, but even then Goldstein's regular $\psi(x,y)$ is not the desired solution. *G. Kuerti* (Cleveland, Ohio)

7464:

Hämmerlin, Günther. Die Stabilität der Strömung in einem gekrümmten Kanal. Arch. Rational Mech. Anal. **1** (1958), 212–224.

The author studies the problem of the stability of a viscous flow in a curved channel. The eigenvalue problem is treated by transforming the differential equations to integral equations which are solved by an iterative technique. The criterion of instability obtained agrees with that given by W. R. Dean [Proc. Roy. Soc. London Ser. A **121** (1928), 402–420] and Reid [ibid. **244** (1958), 186–198; MR **19**, 1119], but disagrees with that given by C. S. Yih and W. M. Sangster [Philos. Mag. (8) **2** (1957), 305–310]. *R. C. DiPrima* (Troy, N.Y.)

7465:

Kovalev, A. A. On Chandrasekhar's spectral representation of axially symmetric turbulence. Soviet Physics. Dokl. **120** (3) (1958), 510–513 (1220–1223 Dokl. Akad. Nauk SSSR).

The problem of turbulence caused by thermal instability has been discussed by Chandrasekhar [Philos. Trans. Roy. Soc. London Ser. A. **244** (1952), 357–384; MR **14**, 328] in an approximation which neglects all third-order moments and treats the turbulence as axisymmetric. With these same approximations, the present author derives the governing equations in spectral form and then reduces them to the equivalent scalar form appropriate for axisymmetric turbulence. A solution is then given for what would correspond to the final period of decay. A formula is also given for the scattering of sound by an axisymmetric turbulent flow.

W. H. Reid (Providence, R.I.)

7466:

Pearson, J. R. A. The effect of uniform distortion on weak homogeneous turbulence. J. Fluid Mech. **5** (1959), 274–288.

In this paper, the effect of a mean rate of strain on homogeneous turbulence is considered from an Eulerian viewpoint. Viscous effects are included; these were difficult to handle in the earlier Lagrangian analyses. The basic assumption is that the mean rate of strain is large compared with the root mean square turbulent rates of strain (i.e. "weak turbulence"), so that linearization of the equations of motion is plausible.

The case of an irrotational rate of strain is considered in detail, and a solution is found for the vorticity spectrum that reduces to the earlier inviscid solutions as the viscosity $\nu\to 0$. Asymptotic expressions at large strains are found for the component intensities in the three cases (1) Two principal axes of contraction, (2) two principal axes of extension, (3) 'constant area' deformation. A discussion is also given of the distortion of turbulence in its final period of decay.

The inclusion of viscous effects in this work removes one of the important limitations in earlier treatments of this problem. *O. M. Phillips* (Baltimore, Md.)

7467:

Wood, Albert D.; and Clarke, Joseph H. An approximate solution for transonic flow in cascades. J. Aero/Space Sci. **26** (1959), 318–319.

Cette courte note traite de l'écoulement transsonique autour d'une grille plane de profils lorsqu'il n'y a ni portance, ni déflection. L'équation non linéaire des petites perturbations est traitée en considérant le terme non linéaire comme provisoirement connu; dans la solution de l'équation de Poisson ainsi obtenue, on évalue le terme provenant des sources, en supposant, pour faire cette évaluation, l'écoulement unidimensionnel. La formule finale pour le coefficient de pression est très simple et la comparaison faite avec un résultat d'expérience est particulièrement satisfaisante. *P. Germain* (Paris)

7468:

Želudev, P. I. Supersonic flow past slender bodies of revolution with and without fins. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1958**, no. 9, 74–82. (Russian)

First, slender-body theory is applied to an inclined, rolled, sharp-nosed body of revolution fitted with slender wings that extend to the base. Appropriate conformal mappings lead to closed expressions for force coefficients

in the case of two or four equally spaced fins, or three fins spaced 90° apart.

Next, a second approximation to slender-body theory is sought by iteration, and the explicit result given for an inclined body of revolution. However, the author mistakenly assumes that the second-order term independent of radius r is related to the term in $\log r$ in the same way as it is in the first approximation. In application to the unyawed cone he accordingly finds disagreement with the known result, which he remedies simply by adding the missing term. *M. D. Van Dyke* (Paris)

7469:

Kleiman, Ia. Z. On the propagation of strong discontinuities in a multi-component medium. J. Appl. Math. Mech. 22 (1958), 268–278 (197–205 Prikl. Mat. Meh.).

The author sets up for each species in a multicomponent medium the equations of conservation of mass and momentum at a moving surface of discontinuity. By contrast with the more familiar conditions in a one component medium, possibilities of the following types arise. A discontinuity may be a shock for some species and a contact discontinuity for others. There may be discontinuity surfaces across which the pressure and the densities of all species remain constant, but there is a flow across the surface and the concentrations of various species are discontinuous. To investigate weak discontinuities the author represents the flow functions behind the discontinuity by their values in front plus small perturbations, and then retains only first order terms in the perturbations. The case of two components is considered in detail. It yields the conclusion that entrainment of a comparatively small amount of gas in a denser liquid reduces the speed of propagation of weak waves to practically the velocity of sound in the gas.

J. H. Giese (Aberdeen, Md.)

7470:

Lun'kin, Iu. P. Entropy change during relaxation of a gas behind a shock wave. Soviet Physics. JETP 34(7) (1958), 1053–1055 (1526–1530 Ž. Eksper. Teoret. Fiz.).

L'auteur calcule la variation d'entropie dans un gaz dans les conditions suivantes: une modification soudaine du gaz sur une longueur de plusieurs libres parcours moyens produit un accroissement de la température qui est dû successivement à la modification de la vitesse de la translation des molécules, de leur vitesse angulaire de rotation, de leurs vitesses relatives (vibrations) et à l'apparition de phénomènes de dissociation. Il calcule dans chaque cas l'augmentation correspondante d'entropie et constate que celle-ci est plus grande dans le cas d'une modification de la vitesse de translation que dans les autres cas. *H. Cabannes* (Marseille)

7471:

Rogers, M. H. Similarity flows behind strong shock waves. Quart. J. Mech. Appl. Math. 11 (1958), 411–422.

Consider one-dimensional, cylindrically- or spherically-symmetrical flow (characterized below by $k=0, 1, 2$) with total energy $E=E_0t^s$. Suppose a strong shock occurs at $r=R(t)$, and let $V=dR/dt$ and $x=r/R$. Then similarity solutions such that $u=Vf(x)$, $\gamma p=\rho_0V^2g(x)$, $\rho=\rho_0h(x)$, which can be generated by a "piston" at $r=x_0R(t)$ where x_0 depends only on k and s, are governed by a system of first order ordinary differential equations for f, g, h. Numerical solutions have been found with an electronic computer for $\gamma=1.2$ and 1.4 with s in the range $0\leq s\leq k+1$. $x_0\,(=f(x_0))$ and $g(x_0)$ have been tabulated, and the partition of E into kinetic and internal energy is discussed.

J. H. Giese (Aberdeen, Md.)

7472:

Šikin, I. S. On exact solutions of equations with shock waves and detonation waves in one-dimensional gas dynamics. Dokl. Akad. Nauk SSSR 122 (1958), 33–36. (Russian)

The equations of ν-dimensional ($\nu=1, 2, 3$) radially symmetrical non-steady flow have solutions of the form

$$t=f(\mu,\nu,A,B),\quad v=(r/\mu)d\mu/dt,$$
$$\rho=\mu^{\nu-1}r^{-1}\varphi'(\mu r),\quad p=\mu r^{\nu}[C+B\varphi(\mu r)],$$

where f is an appropriately chosen function, φ is arbitrary and A, B, C are constants. The author fits such a flow via a shock wave at $r=r_2(\mu)$ to a stagnant gas with constant initial pressure and variable initial density $\rho_1(r)$. The forms of $\varphi(\mu r)$, $r_2(\mu)$, and $\rho_1(r)$ are determined successively by the equations of conservation of momentum, energy, and matter, respectively. Chapman-Jouget detonations are treated similarly.

J. H. Giese (Aberdeen, Md.)

7473:

Galin, G. Ia. Shock waves in media with arbitrary equations of state. Soviet Physics. Dokl. 119(3) (1958), 244–247 (1106–1109 Dokl. Akad. Nauk SSSR).

L'auteur considère un gaz dont l'énergie spécifique interne est une fonction non précisée $E(V, S)$ du volume spécifique et de l'entropie spécifique. Des relations entre cette fonction et la courbe d'Hugoniot (courbe de choc dans le plan pression-volume spécifique) sont établies.

H. Cabannes (Marseille)

7474:

★Officer, C. B. Introduction to the theory of sound transmission, with application to the ocean. McGraw-Hill Series in the Geological Sciences. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1958. viii+284 pp. $10.00.

The considerable expansion of the field of underwater acoustics since the 1940's, and its present status, which is mainly one of great but disorganized activity, has created a very real need for a text outlining both the practical problems and the fundamental theory involved. Although this book does give an idea of what some of the problems are, it fails in the theoretical department. The fraction of the theory that is correctly given is presented in unimaginative fashion, and appears to be merely a restatement of standard treatments that have been given (more elegantly) elsewhere. There is much that is unnecessary, no over-al point of view to provide continuity, and no effort at unification. What is more serious, there is a number of incorrect statements and errors. For example, the criteria of validity of the ray approximation are interpreted as meaning that the variation of sound velocity must be small compared to a wavelength measured along the ray. A simple separation of coordinates, applied to the vector form of the criteria (e.g., in a stratified medium) shows that the pertinent result has the same form as the validity condition for the first order W.K.B. approximation, i.e., it fails at a turning point. The author, being unaware of this fact, is thus led in a later chapter to make the mysterious statement that, in a stratified ocean, "...the ray solution will be valid for frequencies above 3½ cycles per second for an average water depth of 15,000 feet". Granted that under certain conditions predictions can be made by means of the ray theory, even when there are turning points, the above

bald statement is definitely incorrect as it stands. Here is a fundamental point, of central importance to the current theory and practice of underwater acoustics, which is most emphatically not a minor defect of presentation. Another bad error is to be found in the discussion of guided wave propagation in a liquid layer overlying an elastic solid. The author states that the lowest mode has a high-frequency cut-off. Much is made of this erroneous statement and, apart from the obvious offense that it causes to one's physical intuition, it can be traced to the fact that the writer has not noticed that the function $(1-z^2)^{-\frac{1}{2}}\tan(1-z^2)^{\frac{1}{2}}$ is analytic and continuous at $z=1$. This shows the danger of writing books by the technique of "paper-pasting", since this error was originally committed by others in some papers published eleven years ago, the wrong contents of which have been painfully extruded into this book.

To conclude, the reviewer is unable to recommend this book either to students or to professionals. There is too little here to sustain the informed reader's interest, and too much that may mislead the beginner.

I. Tolstoy (Dobbs Ferry, N.Y.)

7475:

Hains, F. D. Some exact solutions to the magnetohydrodynamic equations for incompressible flow. J. Aero/Space Sci. 26 (1959), 246–247.

7476:

Stepanov, K. N. Kinetic theory of magnetohydrodynamic waves. Soviet Physics. JETP 34(7) (1958), 892–897 (1292–1301 Ž. Eksper. Teoret. Fiz.).

E. Åström a montré que les ondes hydromagnétiques dans un gaz ionisé sont les ondes électromagnétiques à basse fréquence rencontrées dans la théorie de la propagation des ondes de radio dans l'ionosphère [Ark. Fys. 2 (1950/51), 443–457; MR 12, 778]. Dans l'article actuel, l'auteur arrive au même résultat par les méthodes de la théorie cinétique; il montre en outre que les ondes qui ne sont pas perpendiculaires au champ magnétique sont amorties et calcule les coefficients d'amortissement.

H. Cabannes (Marseille)

7477:

Resler, E. L., Jr.; and Sears, W. R. The prospects for magneto-aerodynamics—correction and addition. J. Aero/Space Sci. 26 (1959), 318.

Cette note corrige une erreur commise dans un précédent article [J. Aero. Sci. 25 (1958), 235–245; MR 19, 1226]. L'estimation indiquée pour le rapport force électrique/force aérodynamique est à multiplier par 100. On compare les valeurs calculées de la conductivité électrique à celles mesurées par S. C. Lin — les auteurs suggèrent d'utiliser ces dernières, beaucoup plus élevées. Au total, les effets magnéto-aérodynamiques sont encore plus importants que prévu.

J. Naze (Marseille)

7478:

Buneman, O. Transverse plasma waves and plasma vortices. Phys. Rev. (2) 112 (1958), 1504–1512.

On résout les équations aux perturbations relativistes d'un plasma isotrope uniforme comme problème aux valeurs initiales, par transformation de Laplace. En l'absence de tourbillons, les perturbations transverses se propagent par ondes pures, la vitesse de phase étant supérieure à celle de la lumière. Seules les perturbations longitudinales de grande longueur d'onde peuvent persister. Le nombre d'onde critique est calculé pour une distribution Maxwellienne. On étude ensuite la dispersion des tourbillons — il n'y a pas de nouveau mode de propagation.

J. Naze (Marseille)

7479:

Kihara, Taro. Thermodynamic foundation of the theory of plasma. J. Phys. Soc. Japan 14 (1959), 128–133.

Les plasmas au voisinage de l'équilibre thermodynamique sont étudiés sous les hypothèses suivantes: le champ magnétique est faible, la viscosité nulle, on ne prend pas en considération les phénomènes d'ionisation et de recombinaison. De l'équation du mouvement globale et des équations de continuité on déduit l'équation du mouvement pour chaque composant à l'aide de considérations thermodynamiques rigoureuses et par application du théorème de réciprocité d'Onsager. On établit de même l'équation de transfert de chaleur, la loi d'Ohm géneralisée, une équation d'induction dans le cas d'un plasma neutre composé d'électrons et de particules lourdes, et l'équation de diffusion pour un mélange binaire neutre, ainsi que les relations liant les différents coefficients de transfert.

Les résultats obtenus sont appliqués aux plasmas très dilués, aux plasmas de Debye-Hückel, et discutés.

J. Naze (Marseille)

7480:

Kiselev, M. I.; and Tsepliaev, V. I. Oblique shock waves in a plasma with finite conductivity. Soviet Physics. JETP 34(7) (1958), 1104–1106 (1605–1607 Ž. Eksper. Teoret. Fiz.).

L'auteur étudie la structure d'un choc oblique dans un plasma doué d'une conductivité électrique finie; la viscosité et la conductivité thermique sont négligées. L'épaisseur du choc est calculée ainsi que l'angle limite pour la propagation d'un choc oblique dans le cas d'une conductivité électrique infinie; deux valeurs sont obtenues parmi lesquelles une seule est retenue par continuité avec l'aérodynamique. Cette dernière restriction nous semble inutile.

H. Cabannes (Marseille)

7481:

Oganesyan, R. S. Gravitational instability of a plane-parallel layer of conducting fluid in the presence of a magnetic field. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk 11 (1958), no. 4, 39–52. (Russian. Armenian summary)

7482:

Gheorghiţă, Şt. I. Le mouvement lent stationnaire d'un fluide visqueux incompressible dans la présence d'une enveloppe poreuse sphérique. An. Univ. C. "I. Parhon" Bucureşti. Ser. Şti. Nat. 5 (1956), no. 9, 39–45. (Romanian. Russian and French summaries)

The author presents the solution to the problem mentioned in the title, for small Reynolds' numbers (i.e., slow motion; linearized Navier-Stokes' equations). He applies Fourier series expansions for the velocity and pressure components. The ensuing constants are determined from the appropriate boundary-conditions. Particular cases are also noted.

K. Bhagwandin (Oslo)

7483:

Gheorghiţă, Şt. I. Sur le mouvement des eaux artésiennes. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. 6 (1957), no. 15, 51–58. (Romanian. French and Russian summaries)

The author studies the artesian type of ground-water

flow by means of complex-function theory. The media are supposed to be non-homogeneous. The velocity potentials and filtration coefficients satisfy certain conditions at the separation surface. Particular cases are also noted. *K. Bhagwandin* (Oslo)

7484:

Pilatovskiĭ, V. P. Definition and investigation of problems on the stability of shifts of the boundaries between liquids in a heterogeneous filtration system. Ukrain. Mat. Ž. **10** (1958), no. 2, 160–177. (Russian. English summary)

The author presents mathematical analysis of the problem mentioned in the title. The functional equations are solved by means of complex-function theory (singular integral equations), small-parameter asymptotic developments of the Krylov-Bogolyubov type (perturbation theory). Explicit solutions are also presented. For the sake of completeness we reproduce the author's summary below. "The author studies the question of the stability of plane shifts in the boundary of liquid separation in a filtration stream. A system of functional equations was derived determining unsettled shifts in the contact boundary between two liquids. Assuming the known law of motion of a contact boundary, a system of functional equations is derived defining the first variation of the given motion. The resulting equations are applied for investigating the stability of a plane-parallel heterogeneous stream. Criteria of stability and instability are found for the stream. It is shown that these criteria coincide in the case of a stationary process with the "progressing" and "non-progressing" criteria of stratum flooding, which have been obtained earlier by the author on the basis of hydraulic considerations." *K. Bhagwandin* (Oslo)

7485:

Carrier, G. F. The mixing of ground water and sea water in permeable subsoils. J. Fluid Mech. **4** (1958), 479–488.

The author studies subterranean mixing in permeable media of sea water and ground water. The mixing process is studied by means of various models (e.g., C. K. Wenthworth-model, continuum-model, etc.) The physical problem is reduced to the solution of diffusion equations, which are solved in a very elegant manner (i.e., introduction of generating functions, asymptotic evaluation, and Fourier-transform methods). But as the author points out, because of the lack of quantitative information regarding cell size and an exact knowledge of velocity fields in permeable media, it is not possible to describe the salinity distribution in particular regions precisely. It is to be regretted, however, that the author does not refer to other publications dealing with these types of problems. *K. Bhagwandin* (Oslo)

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 7476, 7478, 7479, 7480, 7575.

7486:

★Picht, Johannes. Grundlagen der geometrisch-optischen Abbildung. Hochschulbücher für Physik, Bd. 14. VEB Deutscher Verlag der Wissenschaften, Berlin, 1955. ix+187 pp. DM 25.30.

J. Picht's book is dedicated to the memory of M. Berek, who first suggested a practical way of using first-order aberrations (Seidel theory) for the analysis of practical instruments.

Avoiding the pitfalls and unpermissible simplification of Berek's first attempt, the author gives an elegant introduction to Gaussian optics, first order aberrations and the theory of vignetting.

Ray-tracing formulae are given to trace astigmatism and coma in the tangential plane through any given optical system and formulae are derived in order to analyze the contribution of the single surfaces. [Formulae, which are equivalent to those derived by the referee in Z. Phys. **43** (1927), 750–768, and by F. Cruikshank, Proc. Phys. Soc. **57** (1945), 350–367, 419–434, really do not give the contributions of the single surfaces to the image errors but of the partial system up to the single surfaces.]

A very careful investigation of the asymmetry errors for finite aperture and field of Petzval condition and color aberrations follows.

In the consideration of the Eiconal in the last chapters, the author mainly follows the ideas of a paper by K. Schwarzschild, which are very elegant for third-order aberration but cannot easily be generalized to higher orders.

The book is well written and recommended for the use of lens designers. *M. Herzberger* (Rochester, N.Y.)

7487:

Konjukov, M. V.; and Terletskij, J. P. On the theory of the linear betatron. Nuovo Cimento (10) **9** (1958), 930–941.

This article contains a mathematical discussion of the relativistic motion of an electron in an axially symmetric magnetic field which is moving in the direction parallel to its axis. The Lagrange equations of motion are integrated in three special cases: (1) The motion of the magnetic field is such that the orbit of the electron is a spiral of constant radius; (2) the magnetic field has a constant velocity $u<c$; (3) the magnetic field moves with the velocity of light.

In each case the energy increase is derived in terms of certain assumed magnetic fields, and the corresponding time of flight and path length are calculated approximately. The means for producing localized magnetic fields moving at high velocities are not considered in this paper. *R. D. Kodis* (Providence, R.I.)

7488:

Parker, E. N. Plasma dynamical determination of shock thickness in an ionized gas. Astrophys. J. **129** (1959), 217–223.

Etant donné un gaz raréfié ionisé, l'auteur montre que le phénomène principal qui apparaît dans un choc est une augmentation des oscillations du plasma sous l'effet d'une diminution de l'énergie de translation. Ce résultat étant pris comme base, il calcule l'épaisseur du choc; dans l'espace interstellaire, l'épaisseur trouvée est de l'ordre du kilomètre, le libre parcours moyen étant de l'ordre de 10^8 kilomètres. *H. Cabannes* (Marseille)

7489:

Christiansen, Jens. Über die Kompression einer Plasmasäule im magnetischen Vierpolfeld. Z. Naturf. **13a** (1958), 951–961. (1 plate)

Die Kontraktion einer Plasmasäule in einem zylindrischen Glasrohr, in dem der Plasmazustand durch induktive Ankopplung eines Hochfrequenzfeldes aufrecht er-

halten wird und ausserdem von aussen ein konstantes magnetisches Quadrupolfeld angeregt ist, wird theoretisch untersucht. Das Magnetfeld innerhalb des Glasrohres kann mit Hilfe eines skalaren Potentials beschrieben werden; also haben wir $\mathbf{H}=-\operatorname{grad}\psi$ und $\Delta\psi=0$. Da weiter die Achse des Glasrohres mit der Achse des Quadrupolfeldes zusammenfällt, so lässt sich zeigen, dass der Feldverlauf in einer zu dieser Richtung senkrechten Ebene gut wiedergegeben wird durch die Randbedingungen des Hyperbelfeldes $\psi=\pm\psi_0=\pm\text{const}$ für $y^2-x^2=\pm d^2$, wo d den Halbmesser der Polschuhe bedeutet. Damit folgt endlich für $|x|<d$ und $|y|<d$ das Potential $\psi=\pm\psi_0(y^2-x^2)/d^2$ und weiter $|\mathbf{H}|=2\psi_0(x^2+y^2)^{\frac{1}{2}}/d^2$. Danach wird der Bahnverlauf eines einzelnen geladenen Teilchens in einer Diagonalebene, also für $|x|=|y|$ untersucht. Ein neues Koordinatensystem wird eingeführt und den Ausgangspunkt der weiteren Rechnungen bildet die Lorentzkraft. Der Gedankengang ist in einem gewissen Sinne dem von H. Alfvén [Cosmical electrodynamics, Oxford Univ. Press, New York, 1950; MR **12**, 756] ähnlich, jedoch mit dem grossen Unterschiede, dass hier eben der Bereich $\mathbf{H}\approx 0$ von Interesse ist und deshalb andere Näherungsmethoden angewendet werden müssen. Die erhaltenen Resultate werden besprochen und auch graphisch dargestellt. Danach wird das Problem der Kompression eines ganzen Plasmas berechnet und die Zulässigkeit der dabei gemachten Vernachlässigungen wird besprochen. Zuletzt werden endlich die Berechnungen auch auf alle anderen Ebenen (als die Diagonalebene) näherungsweise erweitert. Als Näherungsformel erhält man, dass das Plasma in dem von den hyperbolischen Zylindern begrenzten Gebiet

$$(1)\qquad x\cdot y=\frac{mvcd}{eH^*},$$

wo $H^*=2\psi_0/d$ ist, vereinigt wird. (Ausser d (Halbmesser der Pohlschuhe) haben in (1) alle Symbole die gewohnte Bedeutung.)

Wegen der weitgehenden Loslösung der Entladung von den Gefässwänden muss diese Methode zur Erzeugung hoher Temperaturen geeignet sein. Noch bessere Resultate könnten mit einem zyklischen Vierpolfeld erreicht werden. Die Methode eignet sich ausserdem auch zur Isotopentrennung. Weitere theoretische und experimentelle Untersuchungen sind im Gange.

T. Neugebauer (Budapest)

7490:

Meyer, F. Untersuchung der Stabilität eines gravitierenden Plasmas in gekreuzten Magnetfeldern. Z. Naturf. **13a** (1958), 1016–1020.

Von M. Kruskal und M. Schwarzschild [Proc. Roy. Soc., London Ser. A **223** (1954), 348–360; MR **15**, 914] wurde das Problem der Stabilität einer Plasmaanordnung untersucht, in der eine gravitierende horizontale Plasmaschicht — in der ein ebenfalls horizontales Magnetfeld vorhanden ist — von einem sich darunter befindenden Vakuummagnetfeld getragen wird und die Kraftlinien der erwähnten zwei Magnetfelder zueinander parallel verlaufen. Die genannten Verfasser fanden eine Instabilität. Die vorliegende Arbeit ist eine Erweiterung der erwähnten Untersuchung, wobei angenommen wird, dass die Richtungen der zwei Magnetfelder miteinander einen Winkel α einschliessen. Aus den Rechnungen folgt eine Stabilisierbarkeit durch endliche Verdrehungswinkel α gegen alle Störungen mit kleinen Wellenlängen.

Den Ausgangspunkt der Berechnungen bilden im Plasma die Bewegungsgleichung, die Kontinuitätsgleichung, die Energiegleichung, das Ohmsche Gesetz und die Maxwellschen Differentialgleichungen. Im Vakuum hat man $\mathbf{B}=\operatorname{rot}\mathbf{A}$, $\mathbf{E}=-\partial\mathbf{A}/\partial t$, $\operatorname{div}\mathbf{A}=0$ und $\Delta\mathbf{A}=0$. ($\mathbf{A}$ ist das Vektorpotential.) An der Grenzfläche wird die Kontinuität der Normalspannungen, der Tangentialspannungen und von der Tangentialkomponente der elektrischen Feldintensität gefordert. Weiter wird angenommen, dass in der Plasmaschicht der Druck, die Dichte und die Intensität des Magnetfeldes mit der Höhe exponentiell abnehmen. Da die Störungen als klein angenommen werden, so kann man die Grundgleichungen in den Störgrössen linearisieren und auf dem Wege folgt ein homogenes lineares Differentialgleichungssystem für die Störungen. Danach wird das Problem der Stabilität mit dem Resultate untersucht, dass für alle Störungen, deren Wellenlängen klein gegen die Schichtdicke sind, die Kreuzung der Magnetfelder um einen Winkel α stabilisierend wirkt. Nur für den Fall, dass die relative Stärke der Störungen langsamer als die erwähnte exponentielle Abnahme abklingt, treten Instabilitäten auf.

T. Neugebauer (Budapest)

7491:

Rawer, K.; und Suchy, K. Statistische Herleitung der Dispersionsformel eines Lorentz-Plasmas endlicher Temperatur. Ann. Physik (7) **2** (1958), 313–325.

Die Verfasser berechnen von neuem die Ausbreitung von elektromagnetischen Wellen in einem Plasma und vergleichen ihre Resultate mit denen von anderen Autoren. Betrachtet wird dabei ausschliesslich ein Lorentz-Plasma, in dem also bloss die Elektronen als beweglich, die (schweren) positiven Ionen dagegen als ruhend angenommen werden. Berechnet werden muss erstens die Mitbewegung der Elektronen durch das Wellenfeld und zweitens dann die Rückwirkung dieser Trägerbewegungen auf das Wellenfeld.

Den Ausgangspunkt des Gedankenganges bildet die bekannte Boltzmann-Gleichung

$$(1)\qquad \frac{\partial f}{\partial t}+\mathbf{c}\cdot\frac{\partial f}{\partial \mathbf{r}}+\frac{\mathbf{K}}{m}\cdot\frac{\partial f}{\partial \mathbf{c}}=\frac{\delta f}{\delta t},$$

in der f die Verteilungsfunktion, $\mathbf{c}$ die Geschwindigkeit, $\mathbf{K}$ die Kraft und m die Masse bedeutet. (1) drückt die Tatsache aus, dass die Änderung von f infolge der Änderungen der Zeit bzw. der Koordinaten und der Geschwindigkeiten durch die auftretenden Stösse (rechte Seite von (1)) wieder rückgängig gemacht werden muss. Bis jetzt hat man den Gradienten-Term $\mathbf{c}\cdot\partial f/\partial\mathbf{r}$ entweder ganz vernachlässigt, oder nur insofern beibehalten, dass man aus (1) durch Mitteilung die Maxwellsche Transportgleichung hergeleitet hat. Die Verfasser berücksichtigen dagegen auch dieses Glied und zeigen, dass man auf diesem Wege eine Dispersionsgleichung für drei Ausbreitungsarten (Hauptpolarisationen) erhält.

Zur Lösung der Boltzmann-Gleichung wird der aus der Elektronentheorie der Metalle bekannte Ansatz

$$(2)\qquad f(\mathbf{r},c,t)=f_{is}(\mathbf{r},c,t)+\frac{\mathbf{c}}{c}\cdot\mathbf{g}(\mathbf{r},c,t)$$

benützt und ausserdem wird angenommen, dass die Zeitabhängigkeit von $\mathbf{E}$ (elektrische Feldstärke) und von $\mathbf{g}$ durch den Faktor $\exp(-i\omega t)$ beschrieben wird, für die Ortsabhängigkeit von $\mathbf{g}$ wird ausserdem ein "Eikonal"-Ansatz gemacht. Mit Hilfe dieser Annahmen und durch Linearisierung der aus der Boltzmann-Gleichung erhaltenen Differentialgleichung wird der auftretende Strom berechnet und mit Hilfe dieses Resultates dann die Wellengleichung gelöst. In der Formel für g tritt ein mit $\mathbf{Y}$ be-

zeichneter Tensor auf, aus den Berechnungen erhält man dagegen $\mathbf{Y}^{-1}$. Um diesen Tensor erstens auf Hauptachsen transformieren zu können, benützen die Verfasser die Lösungsmethode von Cardani und berechnen dann auf diesem Wege die reziproken Eigenwerte. Es sei noch bemerkt, dass nach den Berechnungen in dieser Arbeit der Temperatureinfluss durch einen Parameter charakterisiert wird, welcher den adiabatischen Polytropenexponenten 5/3 enthält.

T. Neugebauer (Budapest)

7492:

Ginzburg, V. L. Electromagnetic waves in isotropic and crystalline media characterized by dielectric permittivity with spatial dispersion. Soviet Physics. JETP 34(7) (1958), 1096–1103 (1593–1604 Ž. Eksper. Teoret. Fiz.).

For the connection between E and D in a medium the author writes

$$(*)\qquad D_i=\varepsilon_{ik}(\omega)E_k+\gamma_{ikl}(\omega)\frac{\partial E_k}{\partial x_l}+\delta_{iklm}(\omega)\frac{\partial^2 E_k}{\partial x_l \partial x_m}.$$

Thus the dielectric behavior of the medium does not only depend on the frequency ω and the direction of a plane wave, but also on its wave-length; this is called spatial dispersion. The resulting dispersion formula is of the third degree, giving rise to three different modes for a given frequency. The properties of these three modes are investigated. In isotropic media and in cubic crystals there are two transverse modes, while the third mode is longitudinal, and is identified with the plasma wave. Energy loss of fast charged particles due to Čerenkov radiation is connected with the longitudinal mode.

{Inasmuch as the relation (*) consists of the first terms of an infinite series, the reviewer hesitates to accept the resulting third-order equation for the wave vector as physically significant.} *N. G. van Kampen* (Utrecht)

7493:

Kahn, F. D. Long-range interactions in ionized gases in thermal equilibrium. Astrophys. J. 129 (1959), 205–216.

In a plasma the electrostatic interactions can be such that the distribution of the charges is not random. Such considerations give rise to the well-known Debye shielding effect. In the present paper the author calculates the autocorrelation function for the whole charge density distribution and its electron component; and the effect of partial ordering on scattering of radiation by electrons.

The charge distribution in a finite volume is represented as the sum of spatial Fourier components. The electrostatic potential energy corresponding to each component is then found in terms of the mean square of the amplitude. For a random distribution of charges at temperature T this energy is inversely proportional to the square of the wave number k. Thus, for modes characterised by small k, the energy can be in excess of the equilibrium value κT. There must therefore be a redistribution of charge to suppress such energies in the case of a gas in equilibrium. The mean square amplitude is recalculated taking such interactions into account, via a recurrence relation obtained from consideration of the change in the distribution when the assembly of charges is increased by one. The same is done for the electrons of the distribution.

When radiation passes through an ionized gas the effect is practically determined by its interaction with the more mobile electrons. The zero-order component of the electron distribution increases the phase velocity and absorbs energy from the beam. The higher-order components are responsible for scattering it in other directions. Approximate formulae for the scattering cross-section per electron are derived. The effect of electrostatic interaction is evidently to reduce the value of the cross-section from that given by the Thomson formula.

K. C. Westfold (Sydney)

7494:

Garibyan, G. M. The theory of transient effects in electrodynamics. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk 11 (1958), no. 4, 7–11. (Russian. Armenian summary)

7495:

Moizhes, B. Ia. On the theory of electromagnetic wave propagation in a helix. Soviet Physics. Tech. Phys. 28(3) (1958), 1196–1201 (1286–1292 Ž. Tehn. Fiz.).

The sheath theory of the helix is refined by utilizing an averaged boundary condition for $E_{\|}$, the electric field parallel to the helix wires, in the case of finite wire spacing. In the region of small spacing, where this boundary condition is valid, the not surprising result is that the correction to the simple sheath theory, which places $E_{\|}=0$, is negligible.

A similar boundary condition is applied to calculate, for a rotationally symmetric mode in a circular waveguide, the attenuation produced by a coaxial helix close to the guide wall. *E. T. Kornhauser* (Providence, R.I.)

7496:

De Wette, F. W.; and Nijboer, B. R. A. The electrostatic potential in multipole lattices. Physica 24 (1958), 1105–1118.

General expressions for the electrostatic potential in perfect multipole lattices are given as expansions in terms of spherical harmonics. The coefficients occurring in these expansions contain lattice sums of a general type, which have been treated previously. Such expressions are derived for a point charge lattice, a multipole lattice, and finally for a lattice built up from a number of different arbitrary charge distributions. *Werner Nowacki* (Bern)

7497:

Battig, A. Formation and propagation of a Cherenkov cone. An. Acad. Brasil. Ci. 30 (1958), 287–294.

The formation of a cone of Cherenkov radiation when an electron enters a dielectric medium from vacuum, and its reflection and refraction on reemerging from the dielectric medium are calculated following a method of G. Beck [Phys. Rev. (2) 74 (1948), 795–802; MR 10, 343].

S. Bludman (Berkeley, Calif.)

7498:

Mertens, Robert. The diffraction of light by superposed parallel supersonic waves: general theory. Proc. Indian Acad. Sci. Sect. A. 48 (1958), 288–304.

The title problem implies scalar diffraction of light by a medium for which the refractive index is of the form

$$\mu(x,t)=\mu_0+\mu_1\sin[2\pi(\nu t-\lambda^{-1}x)] \qquad\qquad +\mu_n\sin[2\pi(n\nu t-n\lambda^{-1}x+\Delta)].$$

Assuming incident light normal to the x-axis, the author derives the difference-differential equations for the diffracted spectra and considers the approximate solution thereof. The results are compared with the less general results obtained by J. S. Murty [J. Acoust. Soc. Amer. 26 (1954), 970–974].

J. W. Miles (Los Angeles, Calif.)

7499:

Mertens, Robert. On the "method of parts" for the diffraction of light by superposed and standing supersonic waves. Simon Stevin 32 (1958), 80–90.

In this article the author treats the propagation of light through two progressive sound beams, of equal wave length moving in opposite direction in a vessel, under the assumption that the effect of the two beams on the light wave is the same as that which results when the two beams are placed in separate adjacent vessels of equal width as in the first case. This hypothesis is based on the experimental evidence obtained by A. Pande, M. Pancholy and S. Parthasarathy [J. Sci. Industr. Res. **3** (1944), 2]. The procedure followed by the author is to divide the width of each vessel into $2N$ equal parts (N being an arbitrary large number) and then consider the contributions to the amplitude of the light emerging from the $2m+2$ cell. By letting the width of the cells approach zero the author has derived, for the amplitude of the $2m+2$ cell, a difference-differential equation which propagates in a certain definite direction different from the initial one. From the boundary conditions the solution of this equation is given in terms of products of Bessel functions whose arguments depend on the width of the vessel (L), the corresponding amplitude of the refractive index of the medium which is perturbed by the sound field and the wave length of the light wave, which become equal when supersonic standing sound waves are considered. The result of this analysis shows that when the order of the spectrum is even, then for two even or two odd spectra the light is partly coherent and for one even or odd spectrum, one has total incoherence. The author states that his analysis, which is based on the ideas (theory) of Raman and Nath [Proc. Indian Acad. Sci. Sect. A **2** (1935), 406–420], is valid for large wave lengths of the sound waves and narrow width of the sound field.

N. Chako (Flushing, N.Y.)

7500a:

Ufimtsev, P. Ia. Secondary diffraction of electromagnetic waves by a strip. Soviet Physics. Tech. Phys. **28** (3) (1958), 535–548 (569–582 Ž. Tehn. Fiz.).

7500b:

Ufimtsev, P. Ia. Secondary diffraction of electromagnetic waves by a disk. Soviet Physics. Tech. Phys. **28**(3) (1958), 549–556 (583–591 Ž. Tehn. Fiz.).

In an earlier paper [Ž. Tehn. Fiz. **27** (1957), 1840–1849; MR **19**, 1012] the author presented a method for the approximate solution of problems of diffraction of plane electromagnetic waves by certain convex bodies with edges. The solution was found by adding to a regular, or uniform, field a correction, called the non-uniform component, obtained by applying the theory of scattering by a wedge to small elements of the surface. The method was applied, in particular, to an infinite plane strip, and it was found that the result failed to represent the field correctly. The author conjectured that the defect could be remedied by taking into account the diffraction by either edge of the strip of the wave from the other edge (secondary diffraction). In the first of the two papers reviewed here the conjecture is confirmed. The secondary diffraction is calculated for either edge by replacing the strip with a half-plane containing the strip and terminating at the opposite edge, then replacing the primary wave by a line source parallel to the edge of and above the half-plane, and finally applying the reciprocity principle. The effect of adding this correction to the earlier result is to obtain an approximate expression for the wave field in the presence of a strip which is satisfactory for all angles of incidence and observation, including grazing angles.

In the second paper the secondary diffraction is calculated for a disk, improving an as yet unpublished earlier solution which again did not account for the interaction of the edge currents. The interaction is approximated by the effect of an elementary dipole above a half-plane. It is shown that the "screening" functions, i.e., those functions of the observation angle in the expressions for the fields which are due to the presence of an edge, are the same for the disk as for the strip. When the interactions are included, the solution thus modified is valid for all angles of incidence. As in the case of the strip, both **E**- and **H**-polarizations are treated.

As the work stands now, a method for computing the field scattered by an object of rather general shape has been presented. For its justification it appeals to the success with which it reproduces known results rather than on rigorous deductions from explicit assumptions. In the two cases for which results have been published, corrections due to secondary diffraction had to be made to achieve the desired success. These corrections appear not to depend critically on the special geometries of the bodies concerned. It will be of interest to see how powerful the method turns out to be when applied to more difficult problems. *R. N. Goss* (San Diego, Calif.)

7501:

Phariseau, P. Diffraction of light by a three-dimensional system of ultrasonics. Physica **24** (1958), 985–995.

Le mémoire traite de la diffraction de la lumière provoquée par les ultra-sons dans une lame de liquide qui est le siège de trois trains d'ondes longitudinales planes. L'absorption de la lumière est négligée, et les oscillations transportées par deux trains d'ondes distincts sont supposées incohérentes. Les plans d'onde des trois trains d'ondes forment, en se coupant, un réseau à trois dimensions, et la variation locale subie par l'indice de réfraction du liquide est en conséquence triplement périodique. Ainsi, le problème posé se ramène à celui de la diffraction des rayons X par le milieu cristallin. L'auteur le résout selon la méthode de Zachariasen [*Theory of X-ray diffraction in crystals*, Wiley, New-York, 1945; p. 115]. Il distingue deux cas: l'un où la lumière diffractée sort du liquide à travers la face d'incidence (réflexion de Bragg), l'autre où la lumière diffractée sort à travers la face opposée à la face d'incidence (réflexion de Laue). Et il trouve, sous certaines conditions, que les trois trains d'ondes peuvent provoquer une réflexion totale, qui a lieu suivant la loi de Bragg. *J. Laval* (Paris)

7502:

Laporte, O.; und Meixner, J. Kirchhoff-Youngsche Theorie der Beugung elektromagnetischer Wellen. Z. Physik **153** (1958), 129–148.

In Kirchhoff's theory, the diffracted field behind an aperture in an infinite screen, due to an incident scalar spherical wave, can be written as a superposition of the geometrical-optics field and the edge-diffracted field. In the paper under review, Rubinowicz's theory [A. Rubinowicz, Acta Phys. Polon. **12** (1953), 225–229] is generalized so as to be applicable to the diffraction of an electromagnetic dipole wave. The authors use tensor analysis to effect the transformation of surface integrals into line integrals.

C. J. Bouwkamp (Eindhoven)

7503:

Heyn, Eugen. Elektromagnetische Felder in gekrümmten Hohlleitern. Abh. Deutsch. Akad. Wiss. Berlin Kl. Math. Phys. Tech. **1958**, no. 4, 41 pp. (1958).

The type of problem with which this paper is concerned is the calculation of the reflexion coefficient when two straight waveguides are joined by a section of constant curvature, the cross section in all three parts being the same. The treatment is approximate in that all powers of the curvature higher than the first are neglected; but, even so, the analysis is too complicated for discussion in a short review. *E. T. Copson* (St. Andrews)

7504:

Wait, J. R.; and Conda, A. M. Pattern of an antenna on a curved lossy surface. Trans. IRE. **AP-6** (1958), 348–359.

This paper concerns the diffraction of a linearly polarized electromagnetic wave $\exp(ikx)=\exp(k\rho\cos\varphi)$ by a circular cylinder ($\rho=a$) with axis in the Z direction. The finite conductivity is accounted for by a proper approximative boundary condition; in the two cases of the magnetic or electric field parallel to the axis the latter amounts to the "Leontovich boundary condition" of the form $\partial\pi/\partial\rho=\gamma\pi$ at $\rho=a$. The expansion of the field into modes $J_n(k\rho)\cos(n\varphi)$ of integral orders n involves a first representation of the "cut-back factor" $F(ka,\varphi)$ that is defined as the ratio of the total field on the cylinder and of the amplitude of the incident field. This first expansion is reduced, with the aid of a "Watson transformation" (well known from the related diffraction problem for a sphere), to another series each term of which is represented by an integral over complex orders n. These terms can be expressed with the aid of Airy functions when substituting the so-called third order or Hankel approximation for the relevant Hankel functions. Further, the successive terms may be interpreted as contributions reaching the receiving point after having travelled a number of times around the cylinder ("creeping waves"). The leading term suffices for large values of ka. The corresponding approximation for F is shown to coincide with that for the attenuation (relative to free space) of the amplitude of the radially directed Hertzian vector that determines the field of an antenna placed on a homogeneous finitely conducting spherical earth; the angular distances from the receiver both to its horizon and to the transmitter then have to be large compared to $(ka)^{-\frac{1}{3}}$.

Numerical computations of F for large values of ka are represented by various diagrams. They are largely based on the methods known from the conventional theory for the diffraction of radio waves around the spherical earth. In fact, a geometric-optical method is applied in the region lit by the incident wave, whereas a "residue series" derived from the above mentioned leading term proves to be convenient in the shadow region. The transition zone between these two regions is bridged by direct numerical evaluation of the integral in question. The introduction of proper parameters permits an application to the diffraction by a cylinder or sphere of any conductivity and dialectric constant. Additional numerical results for $ka=8$ show the reasonable accuracy of approximations for moderate ka values that are based on the two first terms of the expansion with respect to creeping waves. It is suggested that the numerical results should also apply to other smooth surfaces if a is replaced by some principal radius of curvature of the latter.

H. Bremmer (Eindhoven)

7505:

Kim, Wan Hee. Topological evaluation of network functions. J. Franklin Inst. **267** (1959), 283–293.

Certain topological expressions of a network not containing coupled coils are defined in terms of the branch admittances. The necessary conditions are derived for the physical realizability of three and four terminal networks which are expressed in terms of the topological expressions. Networks with a large number of elements can be more readily handled by decomposing them into simple sub-networks whose topology can be examined by inspection. Means for determining the desired topological expressions of the network by various combinations of the sub-networks are given.

R. Kahal (Washington, D.C.)

7506:

Lindgren, B. W. On Samulon's formula for frequency response from step response. SIAM Rev. **1** (1959), 47–49.

A new derivation is given of a formula by Samulon [Proc. I. R. E. **39** (1951), 175–186; MR **12**, 447] for the frequency response of a linear system whose frequency response vanishes above a given frequency, based on a discrete sampling of the unit step function transients. A means for estimating the error due to the non-zero response above the assumed cut-off frequency is also given. *R. Kahal* (Washington, D.C.)

7507:

Gould, Roderick. Graphs and vector spaces. J. Math. Phys. **37** (1958), 193–214.

In an abstract graph with k arcs, any subset of the k arcs can be represented by a vector ξ with k components in the field of the integers modulo 2, where components 1 correspond to arcs in the subset and components 0 correspond to arcs not in the subset. Thus one gets an arc space. If a loop-set is a set of arcs whose branches (union of arc with incident vertices) form a loop or a union of arc-disjoint loops, then the collection of loop-sets determines a vector space, the loop-set subspace of the arc space. Given an arbitrary matrix M over the integers modulo 2, a systematic procedure is developed for finding a graph, if one exists, whose loop-set space is the vector space spanned by the rows of M. This is of interest in the synthesis of contact networks.

S. Sherman (Philadelphia, Pa.)

7508:

Abdel-Messih, Moheb Aziz. Zeros and poles of output voltage of 3-terminal potentiometer networks. Z. Angew. Math. Phys. **10** (1959), 207–215. (German summary)

Consider a three terminal network containing only fixed resistors and a finite number of linear potentiometers mounted on a common shaft. Let x represent the angular displacement of the shaft from one end of the potentiometer winding expressed as a fraction of its full range; $0\leqq x\leqq 1$. If a constant voltage E_1 is applied to the input terminals, then the output voltage $E_2(x)$ is a function of x. The following theorem is proved. The output voltage $E_2(x)$ is a rational function of x with the following properties: (i) the zeros of $E_2(x)$ are either real outside the interval $0<x<1$, or occur in conjugate complex pairs; and (ii) the poles of $E_2(x)$ are simple real negative or >1.

R. Kahal (Washington, D.C.)

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See also 7439, 7455.

7509:

Hart, Edward W. Thermodynamics of inhomogeneous systems. Phys. Rev. (2) **113** (1959), 412–416.

"A self-consistent thermodynamic formalism is developed for the treatment of the equilibrium of systems, some of whose parameters vary continuously from place to place. The method is specially designed for the description of transition interfaces separating two phases. The energy per unit volume is assumed to depend explicitly on the space derivatives of the molecule densities. Equilibrium conditions are obtained for the appropriate internal variables of the system, and all externally measurable intensive variables are uniquely defined by a variational procedure." (Author's summary)

W. Byers Brown (Manchester)

7510:

Kim, E. I. On a heat conduction problem for a system of bodies. Prikl. Mat. Meh. **21** (1957), 624–633. (Russian)

The paper discusses the two-dimensional problem of thermal conduction in two media of different thermal diffusivity (a_1 and a_2) which are in perfect thermal contact along a straight line. The author states that the problem cannot be solved by iteration. The problem is reduced to an integral equation which is solved by the application of a Fourier and a Laplace transform and the existence of the solution is proved. *J. Kestin* (London)

7511:

v. Krzywoblocki, M. Z.; and Bloomquist, R. E. On heat phenomena on rotating bodies in free molecule flow. Acta Phys. Austriaca **12** (1958/59), 237–245.

"The authors derive the formulas and discuss the heat transfer phenomena on rotating spheres and cylinders in free-molecule flow of a monatomic gas. Radiation effects are neglected." (From the authors' summary)

D. W. Dunn (Ottawa, Ont.)

7512:

Prudnikov, A. P. Analytic investigation of processes of heat and mass transfer in convective drying. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1958**, no. 10, 63–67. (Russian)

7513:

Sestini, Giorgio. Problemi di propagazione del calore con convezione forzata. Riv. Mat. Univ. Parma **8** (1957), 5–14.

The physical problem of the title is mathematically equivalent to the parabolic equation

$$k\Delta U - \mathbf{v}\cdot\operatorname{grad} U = \frac{\partial U}{\partial t} + A + cU$$

for $t>0$ in some region S of space, given that U satisfies an initial condition $U=f\ (t=0)$ and a boundary condition $a(\partial U/\partial n)+bU=\varphi$ on the boundary of S. Here k, a, b are constants, f a given function of the cartesian coordinates (x, y, z) and A, c, φ given functions of (x, y, z, t).

If the solution W of the corresponding problem for the reduced equation (i.e., the same equation with $\mathbf{v}$, A, c identically zero) is known, it can be shown that U satisfies an integro-differential equation involving W and the Green's function of the reduced equation. And, by integration by parts, this integro-differential equation can be turned into an integral equation which can, in theory, be solved by successive substitutions.

The author applies this theory to two problems relating to the half-space $x\geqq 0$. These are: (a) a one-dimensional problem, in which U is a function of (x, t) alone and satisfies

$$k\frac{\partial^2 U}{\partial x^2} - v(x,t)\frac{\partial U}{\partial x} = \frac{\partial U}{\partial t},$$

$$U=\alpha x \quad (x>0,\ t=0),$$

$$U=-\lambda t \quad (x=0,\ t>0);$$

(b) a three-dimensional problem in which

$$k\Delta U - v(x,t)\frac{\partial U}{\partial x} = \frac{\partial U}{\partial t},$$

$$U=\alpha x+\beta y+\gamma z \quad (x>0,\ t=0),$$

$$U=-\lambda t \quad (x=0,\ t>0).$$

For (a) he carries out the first five steps in solving the integral equation by successive substitutions, in the case when v is of the form $m(x)p(t)$; but for (b) he does not get beyond the first approximation.

E. T. Copson (St. Andrews)

7514:

Hunziker, Raul R. Heat transfer and Ryenolds' analogy in a turbulent flow with heat release. Z. Angew. Math. Phys. **9a** (1958), 307–315. (German summary)

The paper shows that the Reynolds analogy may be extended to a fully developed turbulent flow in a circular pipe with constant wall temperature and heat generation if the heat source intensity is proportional to the constant axial pressure gradient and the product of Pr and α is unity, where Pr is the Prandtl number and α the ratio of eddy diffusivity for heat to that for momentum.

L. N. Tao (Chicago, Ill.)

7515:

Tirskiĭ, G. A. Distribution of temperature in a non-homogeneous rod of variable cross-section in a gas flow. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 1, 70–76. (Russian)

Approximate solutions of the differential equation governing the problem

$$\frac{d}{d\xi}\left[f(\xi)\frac{d\theta(\xi)}{d\xi}\right] - \varepsilon^{-2}g(\xi)\theta(\xi) = -\varepsilon^{-2}g(\xi)\theta_\Gamma(\xi)$$

are obtained as exact solutions of auxiliary differential equations. Detailed numerical examples are worked out with different forms of the coefficients $f(\xi)$, $g(\xi)$ and $\theta_\Gamma(\xi)$, and the errors are evaluated.

C. D. Calsoyas (Livermore, Calif.)

7516:

Kapur, J. N. A note on the solution of the equations of internal ballistics for the general linear law of burning. Proc. Nat. Inst. Sci. India. Part A **24** (1958), 226–229.

The author, having previously given four methods of solving the equations of internal ballistics when the rate of burning is a linear function of the pressure, here offers a fifth method, simpler but less flexible than the others, again using dimensionless units.

A. A. Bennett (Providence, R.I.)

QUANTUM MECHANICS

See also 7221, 7337, 7458, 7561, 7562, 7570, 7572.

7517:

Dalgarno, A. Application of the Rayleigh Schrödinger perturbation theory to the hydrogen atom. Proc. Phys. Soc. Sect. A. **69** (1956), 784–785.

The author shows that second order perturbation theory in $2(Z-Z_1)/r$ for the ground state energy of the equation

$$\{\nabla^2+2Z_1r^{-1}+2(Z-Z_1)r^{-1}\}\Psi=\varepsilon\Psi$$

yields the exact result when $Z_1\neq 0$. This had been conjectured by R. Trees [Phys. Rev. (2) **102** (1956), 1553–1556].
A. S. Wightman (Princeton, N.J.)

7518:

Hanke, L.; und Urban, P. Zur Greenschen Funktion der Diracgleichung. Acta Phys. Austriaca **12** (1958/59), 304–314.

This paper, summary of a doctoral dissertation by the first-named author, expresses the Green's function of a Dirac electron in the presence of a homogeneous magnetic field ($\mathbf{B}$=const), both in physical space-time as an expansion in Hankel functions, and in energy-momentum (Fourier) space in terms of the Whittaker function.
P. G. Bergmann (New York, N.Y.)

7519:

Schwartz, Charles. Calculations in Schrödinger perturbation theory. Ann. Physics **6** (1959), 156–169.

The first order perturbed equation of Rayleigh-Schrödinger perturbation theory is solved analytically for a number of perturbations of the hydrogen atom in the ground and metastable states. *A. Dalgarno* (Belfast)

7520:

Rarita, William; and Schwed, Philip. Minimum theorem for the interaction radius in two-body collisions. Phys. Rev. (2) **112** (1958), 271–272.

It is shown that the phase shift analysis of a two body scattering process must incorporate at least l non-negligible phase shifts when l is determined by $(l+1)^2\geqq k^2\sigma_t^2/4\pi\sigma_e$, where σ_t is the total cross-section, σ_e the elastic scattering cross-section and k the wave number. It is also shown that, for σ_e to be a minimum for a given σ_t, the phase shift is pure imaginary if $\sigma_e\leqq\sigma_t/2$ and has a real part equal to $\pi/2$ if $\sigma_e>\sigma_t/2$.
H. Feshbach (Cambridge, Mass.)

7521:

Born, Max. Vorhersagbarkeit in der klassischen Mechanik. Z. Physik **153** (1958), 372–388.

Several examples in classical mechanics are used to demonstrate that a small uncertainty in the initial conditions may give rise to a large uncertainty in the state of a system after a long time. Hence complete determinism in the sense of Laplace does not exist even in classical physics. It is argued that this remark obviates some of the philosophical objections to quantum mechanics.
N. G. van Kampen (Utrecht)

7522:

Harris, Joseph David. Green's functions for particles of arbitrary spin. Phys. Rev. (2) **112** (1958), 2124–2126.

The author considers the Green's functions associated with the general relativistic wave equation for particles of arbitrary spin [H. J. Bhabha, Rev. Mod. Phys. **17** (1945), 200–216; MR **7**, 272]. Their expression in terms of standard invariant functions is determined by the properties of the appropriate matrix representation of the proper Lorentz group and, in particular, by the form of the minimal equation satisfied by these matrices. The author points out that if there are three or more different non-degenerate masses associated with the representation, then the Green's functions are not singular on the light cone. He does not remark that this is achieved at the cost of introducing an indefinite metric.
J. C. Polkinghorne (Cambridge, England)

7523:

Majumdar, S. Datta. The Clebsch-Gordan coefficients. Progr. Theoret. Phys. **20** (1958), 798–803.

For a dynamical system with two angular momenta, M_1 and M_2, the normalized spherical harmonics $Y_j^m(\Theta,\varphi)$, which simultaneously diagonalize $M_z=M_{z_1}+M_{z_2}$ and the total angular momentum $M^2=(M_1+M_2)^2$, are linear combinations of products of the type

$$Y_{j_1}^{m_1}(\Theta_1,\varphi_1)Y_{j_2}^{m_2}(\Theta_2,\varphi_2).$$

The coefficients $\{m_1m_2|jm\}$ of these linear combinations are called the Clebsch-Gordan coefficients of the first kind. If the normalized eigenfunction Ψ_{jm} of the operator M^2 is multiplied by $[(j-m)!(j+m)!]^{\frac{1}{2}}$, the spherical harmonics are replaced by $e^{im\varphi}$ and the usual angular momentum operators by the operators

$$M_x\pm iM_y=e^{\pm i\varphi}(j\pm iD_\varphi),\ M_z=-iD_\varphi,\ D_\varphi=\frac{d}{d\varphi},$$

then the author gets the function

$$\Phi_{jm}=\sum_{m_1m_2}(m_1m_2|jm)\exp(im_1\varphi_1+im_2\varphi_2).$$

The coefficients $(m_1m_2|jm)$ of this double Fourier series are called Clebsch-Gordan coefficients of the second kind. The author proves the relations

(a) $\Phi_{jm}=e^{im\xi}\sum_{m_2}(m_1m_2|jm)e^{-im_2\eta}=e^{im_2\xi}\chi_{jm}$ $\quad(\xi=\varphi_1,\ \eta=\varphi_1-\varphi_2)$,

(b) $[e^{i\eta}(j_1-m+iD_\eta)(j_2+iD_\eta)+e^{-i\eta}(j_1+m-iD_\eta)(j_2-iD_\eta)+2(im+D_\eta)+F_1+F_2-F]\chi_{jm}=0$ $\quad(D_\xi=\partial/\partial\varphi_1,\ D_\eta=\partial/\partial(\varphi_1-\varphi_2))$,

(c) $[j_1-m+iD_\eta+e^{-i\eta}(j_2-iD_\eta)]\chi_{jm}=(j-m)\chi_{jm+1}$,

(d) $[j_1+m+1-iD_\eta+e^{i\eta}(j_2+iD_\eta)]\chi_{jm+1}=(j+m+1)\chi_{jm}$.

(b) determines the functions χ_{jm}, and therefore the Clebsch-Gordan coefficients, up to an arbitrary factor involving j_1, j_2, j, m. Putting $x=e^{-i\eta}$ the author gets a connection of the equations (a), $\cdots$, (d) with the hypergeometric function and general formulae for the Clebsch-Gordan coefficients. *M. Pinl* (Cologne)

7524:

Nagy, K. L. On a possibility for the elimination of the non-physical consequences of the indefinite metric. Nuovo Cimento (10) **10** (1958), 1071–1077. (Italian summary)

"Si è analizzato il metodo di Bogoljubov per l'eliminazione delle conseguenze non fisiche della metrica indefinita. Abbiamo dimostrato che questo metodo è equivalente a una teoria non locale molto complicata, in cui, tuttavia, non occorre far uso della metrica indefinita e sono applicabili i concetti usuali di una teoria quantistica. Abbiamo inoltre delineato il modo di costruire una teoria non locale dotata di relazioni di commutazioni ordinarie."
Riassunto dell'autore

7525:

Hellman, Olavi. On the solution of the one dimensional Schrodinger equation in case of a potential well. Ann. Acad. Sci. Fenn. Ser. A. VI. no. 11 (1958), 9 pp.

The author translates into vector-matrix terms an iterative solution of the equation within the well, and the technique of matching this with suitable exponential solutions valid outside the well. For his earlier work on the case of a finite interval see Z. Angew. Math. Mech. **35** (1955), 300–315 [MR **17**, 489].

F. V. Atkinson (Canberra)

7526:

Lenard, Andrew. Adiabatic invariance to all orders. Ann. Physics **6** (1959), 261–276.

"The purpose of this article is to extend adiabatic invariance theorems to all orders in the slowness parameter. The concept of adiabatic invariance to all orders is formulated precisely in section I. In section II, a classical one-dimensional nonlinear oscillator is treated. It is proved that the action integral extended over a period of the instantaneous time independent problem is an adiabatic invariant to all orders. Section III is devoted to the extension of the Born-Fock quantum mechanical adiabatic theorem to all orders. The systems to which the proof applies in all rigor are those which have a finite number of nondegenerate quantum states which do not cross in the process of adiabatic change.

The proofs are based on a systematic construction of the asymptotic expansion in the slowness parameter of the statistical distribution function which describes an initially stationary ensemble of systems."

Author's summary

7527:

✱Destouches, Jean-Louis. Corpuscules et champs en théorie fonctionnelle. Les Grands Problèmes des Sciences. IX. Gauthier-Villars, Paris, 1958. vii+163 pp. 4000 francs.

In a previous monograph [*La quantification en théorie fonctionelle des corpuscules*, Gauthier-Villars, Paris, 1956; MR **19**, 213] the author developed a theory in which a particle is described by a function u of space and time. By allowing u to have more than one component he now treats the non-relativistic spin $\frac{1}{2}$ particle, and the Dirac particle with and without isotopic spin. By joining two Dirac particles a photon is constructed and it is suggested that the nonlinear terms are responsible for keeping both "demi-photons" together. From this arises a nonlinear electromagnetic field. Similarly, particles with spin 2 are constructed, giving rise to another nonlinear field, which the author identifies with the gravitational field.

N. G. van Kampen (Utrecht)

7528:

Lee, Benjamin W. Dispersion relation for nonrelativistic potential scattering. Phys. Rev. (2) **112** (1958), 2122–2124.

A new dispersion relation is derived for scattering by a nonrelativistic potential of finite extent. The derivation uses contour integration along a semicircle in the lower half of the complex k plane. The residues in this region come from virtual states and radioactive decaying states. Connection with the Breit-Wigner formula is discussed and it is shown that this formula follows correctly from the analytic properties of the S matrix. Explicit expressions are given for the relationship between the resonance energy and half width on the one hand, and the real and imaginary parts of the complex pole of the S matrix, on the other.

M. J. Moravcsik (Livermore, Calif.)

7529:

Sims, A. R. Certain aspects of the inverse scattering problem. J. Soc. Indust. Appl. Math. **5** (1957), 183–205.

The problem considered here is the determination of potentials or indices of refraction which will give rise to a given reflection coefficient. The sufficient conditions that an analytic function be a reflection coefficient have been given by Kay [Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-74** (1955); MR **16**, 1113]. In the paper being reviewed the necessity of these conditions is shown. The paper is marred by a number of misprints.

H. Feshbach (Cambridge, Mass.)

7530:

Faddeev, L. D. Dispersion relations in non-relativistic scattering theory. Soviet Physics. JETP **35(8)** (1959), 299–303 (433–439 Ž. Eksper. Teoret. Fiz.).

It is proposed that the dispersion relations for the scattering amplitude for potential scattering be deduced by studying the Green's function of the total Hamiltonian. The method is illustrated for the case of a three-dimensional non-relativistic equation for scattering from a fixed center. The relations previously obtained by Wong [Phys. Rev. (2) **107** (1957), 302–306; MR **19**, 364] and Khuri [ibid. **107** (1957), 1148–1156] are derived rigorously. The existence of dispersion relations is related to the problem of a complete description of the S matrix for various potential scattering equations.

Author's summary

7531:

Freeman, A. J. Compton scattering of x-rays from nonspherical charge distributions. Phys. Rev. (2) **113** (1959), 169–175.

The Compton scattering of X-rays can be approximated in terms of form factors f_{jk} of the type

$$(1)\qquad f_{jk}=\int \psi_j^*\psi_k \exp(i\kappa\mathbf{S}\cdot\mathbf{r})dv,$$

where the ψ's are appropriate one-electron wave functions; $\kappa/2\pi$ is the wave number of the incident radiation; and $\mathbf{S}=\mathbf{s}-\mathbf{s}_0$ such that $\mathbf{s}$, $\mathbf{s}_0$ are unit vectors along the reflected and incident directions. For nonspherical charge distributions, the scattering cannot be determined as usual merely by computing (1) with $\mathbf{S}$ set along the polar axis, but must be calculated for an arbitrary orientation of $\mathbf{S}$. The problem of integrating (1) under these conditions is dealt with by rotating the ψ to a new coordinate frame (i.e., transformation of their spherical harmonic component) such that $\mathbf{S}$ is once again parallel to the polar axis.

"Application is made to derive the one-electron scattering matrix elements from atoms with s, p and d electrons. It is shown that by a proper averaging over all orientations of the scattering vector, "mean" scattering formulas result which may be used directly for predicting the scattering from monatomic gases."

T. Erber (Chicago, Ill.)

7532:

Yamamoto, Kunio. Theory of unstable particles in the wave-packet-formalism. Progr. Theoret. Phys. **20** (1958), 857–867.

"Scattering problems for finite time intervals are discussed with the aid of a wave-packet-formalism. A pre-

scription is given which allows us to deal with observations of different wave packets in an antinomic way. The unstable particle is investigated in connection with transition probabilities in the framework of the wave-packet-formalism." (Author's summary.)

G. Källén (Lund)

7533:

Gross, E. P. Classical theory of boson wave fields. Ann. Physics 4 (1958), 57–74.

By studying the associated semiclassical theory, a scalar non-relativistic field is quantized which is both non-local and non-linear. The interaction Hamiltonian is $\frac{1}{2}\int\psi^+(x)\psi^+(y)V(x-y)\psi(x)\psi(y)d^3xd^3y$ and therefore represents a two-body interaction. Various classes of exact solutions are investigated. One class has a space independent field as ground state (uniform solution). Excited states describe large amplitude traveling waves, and small amplitude disturbances which satisfy the Bogolyubov dispersion relation. If the potential V is sufficiently attractive there exist (spatially) periodic solutions whose ground state is below that of the uniform solution. The corresponding small amplitude excitations are studied. The paper concludes with a semiclassical theory of the motion of foreign atoms in the boson fluid.

F. Rohrlich (Baltimore, Md.)

7534:

Nagy, K. L. Free field operators and the Yang-Feldman formalism. Acta Phys. Acad. Sci. Hungar. 9 (1958/59), 269–274. (Russian summary)

The commutation relations of the fields associated with the Yang-Feldman formalism and the expressions for scattering amplitudes in terms of them are rederived by simple methods.

J. C. Polkinghorne (Cambridge, England)

7535:

Dyson, Freeman J. Integral representation of a double commutator. Phys. Rev. (2) 111 (1958), 1717–1718.

This note is a contribution to the representation problem for vacuum expectation values, i.e., the problem of parametrizing those temperate distributions which can appear as the vacuum expectation values of products (or retarded products or time ordered products) of local relativistic fields. The author considers

$$D(y,z)=D_{CBA}(y,z)=\langle[C(x_3),[B(x_2),A(x_1)]]\rangle_0,$$

where $y=x_1-x_2$, $z=x_2-x_3$, A, B and C are three scalar fields and x_1, x_2, x_3 are space time points. He obtains the representation

$$(1)\quad D(y,z)=\int_0^\infty ds\int_0^\infty dt\int_0^1 d\lambda\psi(s,t,\lambda)\Delta_s(y)\Delta_t(z+\lambda y).$$

The proof uses the representation formula of Jost and Lehmann [Nuovo Cimento (10) 5 (1957), 1598–1610; MR 19, 1014] together with an elementary identity connecting the sine function to the Bessel function of imaginary argument. It is not stated precisely what classes of D and ψ are permitted. That this is not a pure formality is indicated by the fact that simple examples in quantum field theory are not of the form (1). (Remark due to H. Araki.) For example, take

$$A(x)=\phi_2(x)\phi_3(x),\ B(x)=\phi_1(x)\phi_3(x),\ C(x)=\phi_1(x)\phi_2(x),$$

where ϕ_1, ϕ_2, ϕ_3 are three commuting free scalar fields. Then the quantity which would be indicated by $\int_0^\infty ds\cdot\psi(s,t,\lambda)\Delta_s(y)$ in (1) increases exponentially in y so that no ψ exists.

A. S. Wightman (Princeton, N.J.)

7536:

Arnowitt, R.; and Deser, S. Quantum theory of gravitation: general formulation and linearized theory. Phys. Rev. (2) 113 (1959), 745–750.

The authors consider the problem of quantizing general relativity using the Schwinger action principle [J. Schwinger, same Rev. 82 (1951), 914–927; 91 (1953), 713–728; MR 13, 520; 15, 81]. In general relativity, as in electrodynamics, the existence of function-type group invariances gives rise to constraint relations between the canonical variables. The authors point out that the search for the independent canonical variables of such a system (which in the quantum theory satisfy the usual canonical commutation relations) is facilitated by casting the field equations in first order form and constructing the associated action integral in the form described in Schwinger's second paper. They therefore discuss the Palatini formulation of general relativity, in which the ten components of the metric tensor density $\mathfrak{G}^{\mu\nu}$ and the forty components of the affinity $\Gamma^\alpha_{\beta\gamma}$ are treated as independent field variables. They show that by combining these quantities into a fifty component field χ one transforms the Lagrangian density $\mathfrak{G}^{\mu\nu}R_{\mu\nu}$ (after adding a suitable divergence) into the standard Kemmer form [N. Kemmer, Proc. Roy. Soc. London Ser. A 173 (1939), 91–116; MR 1, 95] for a massless spin two field with a certain cubic interaction term. In this formalism the operator ordering problem becomes trivial, since different symmetrizations of the cubic term lead to the same Hermitian quadratic structure when c-number variations $\delta\chi$ are performed.

In the remainder of the paper the linearized theory of gravitation is worked out in detail according to the scheme, as a preliminary to a treatment of the full gravitational theory. Here the cubic term reduces to a quadratic form, since the self-interaction of the field has been eliminated. The authors show that the constraints are of two types: algebraic constraint equations which may be dealt with trivially, and differential constraint equations which have to be solved by using the properties of the group of transformations. The elimination of these constraint variables leads to the definition of the "radiation gauge" in the theory of gravitation.

P. W. Higgs (London)

7537:

Gotō, Tetsuo. On the unstable states in quantum field theory. Progr. Theoret. Phys. 21 (1959), 1–17.

Recently, the discussion of unstable particles within the framework of the Lee model has become a very fashionable subject. The paper under review here adds to the subject a very careful discussion of the boundary value problem of the Schrödinger equation. It is shown that one can construct a non-normalizable solution of the Schrödinger equation which has a complex eigenvalue for the energy. The imaginary part of the energy is the lifetime of the unstable particle. This state does not belong to the conventional Hilbert space of the incoming particles as it has an infinite number of V-particles at $t=-\infty$. The connection between the "stationary" treatment of this kind and a more conventional treatment of the decaying particle as a time dependent problem with wave packets of normalizable states in ordinary Hilbert space is also discussed. The results are identical in the two cases. The whole problem is very similar to a discussion of decaying levels in nuclear physics.

G. Källén (Lund)

7538:

Glaser, V.; Lehmann, H.; and Zimmermann, W. Field operators and retarded functions. Nuovo Cimento (10) 6 (1957), 1122–1128.

This paper continues the series on the general formulation of quantized field theories which was begun in two previous papers by Lehmann, Symanzik and Zimmermann [Nuovo Cimento (10) 1 (1955), 205–225; 6 (1957), 319–333; MR 17, 219; 19, 1133]. It is proved that when a field operator $A(x)$ is expanded with respect to products of the ingoing field $A_{in}(x)$, the coefficients in the expansion are the retarded functions (vacuum expectation values of retarded products of $A(x)$) introduced in the earlier papers. They satisfy a system of equations which turns out to be a sufficient as well as a necessary condition for such functions to yield a local field operator.

The results described here are valid only for theories without stable bound states. The more general case has since been treated by Nishijima [Prog. Theor. Phys. 17 (1957), 765–802; Phys. Rev. (2) 111 (1958), 995–1011; MR 19, 502; 20 #3007]. *P. W Higgs* (London)

7539:

Taylor, John G. Dispersion relations and Schwartz's distributions. Ann. Physics 5 (1958), 391–398.

The equivalence between strict causality and dispersion relations for a system with an output which is a linear time-independent function of the input is shown to hold under the very general condition that the time-delay distribution $A(t)$ is a tempered distribution in the sense of Schwartz. This result is necessary for a rigorous discussion of the dispersion relations in quantum field theory and in particular is used to prove the theorem of "boundedness in the edge of the wedge" which has been used in that proof, and also in discussions of causality in quantum field theory. *G. Temple* (Oxford)

7540:

Bates, D. R. Electron capture in fast collisions. Proc. Roy. Soc. London. Ser. A 247 (1957), 294–301.

The usual formulation of electron capture processes by fast positive ions depends upon an interaction potential, which is to some extent arbitrary. By using a new expansion of the total wave function, this defect is overcome. Both the impact parameter and wave treatments are presented. *A. Dalgarno* (Belfast)

7541:

★Endt, P. M.; and Demeur, M. (Editors) Nuclear reactions, Vol. I. Series in Physics. North-Holland Publishing Co., Amsterdam; Interscience Publishers Inc., New York; 1959. xii+502 pp. $12.50.

With the extensive effort which has gone into the realm of low energy nuclear physics it has become increasingly difficult for any one individual to have a complete grasp of the entire field from both the experimental and theoretical point of view. While this volume, which is the first of two, does not attempt to fill this void, it does present a number of capable authors each writing in a field in which he is particularly well-versed. None of these authors has attempted, for good reason, to give a complete bibliography of the literature in his specialty, nor has he attempted to deal with every facet of the area in which he is writing. The end result is that, by and large, each chapter is a rather complete entity describing the individual author's own point of view, but, from the point of view of the complete book, there is a general lack of coherence. In all probability this is more a reflection of the field in which the authors are writing, rather than of any lack of editorial supervision and rewriting. Almost all of the articles give a reader who is only slightly familiar with the details of a particular field enough background and detailed information, as well as references, to enable him to start activity in that field should he so desire. It is the reviewer's opinion that the editors might have presented a more logical and clear picture of nuclear reactions by more closely integrating the various chapters. For example, Kerman's chapter on nuclear rotational motion might have been presented with the chapter on Coulomb excitation which will be in vol. II, and Gugelot's survey article on medium energy experimental results, as well as the LeCouteur contribution on the statistical model, might have been presented with the chapter, in vol. II, on direct interactions.

While not attempting either to review or to describe in detail the various contributions it would be of value simply to list the authors and subject material. R. V. Eden has written a brief, lucid, introductory article on "The nucleus as a many-body system". J. P. Elliot has described the "Shell model" and A. K. Kerman has described "Nuclear rotation motion", these two articles clearly supplementing each other. Resonance reactions have been described from both the point of view of their basic physical significance and the formal description of resonance reactions in two companion articles by Erich Vogt and H. E. Gove. Dr. Gove's articles present a large number of curves and generally useful data enabling the experimentalist to reduce his raw data to the quantities necessary to describe the physics of the reaction. D. J. Hughes and R. L. Zimmerman have taken one particular aspect of resonance reactions, namely low energy "Neutron resonances in heavy nuclei", and presented a rather complete and very well written article. Necessary to a complete description of nuclear reactions are particularly those quantities whose observation is more or less independent of the dynamics of the nuclear reaction but reflect the general conservation laws. L. J. B. Goldfarb considers the results of such conservation laws in his chapter on "Angular correlation and polarization". The chapters, by J. H. Fremlin on "Heavy ion reactions", by P. C. Gugelot presenting "A survey of scattering and reaction experiments with medium-energy nucleons and alpha-particles" and by K. J. LeCouteur on "The statistical model", all present a fair amount of experimental data and theoretical interpretation of these data, wherever possible. The particular advantage of these last three chapters is that it enables the reader to see where the general level of understanding in these particular aspects of nuclear physics is at present. *N. S. Wall* (Cambridge, Mass.)

7542:

Villi, C. On the momentum dependence of the nuclear potential. Nuovo Cimento (10) 10 (1958), 259–291. (Italian summary)

"It is shown that the potential energy of a nucleon in the nucleus, evaluated in the first approximation of the perturbation method or on the basis of Brueckner's theory, obeys a hyperbolic partial differential equation, which is independent of any nuclear parameter and is established by the antisymmetry properties of the nucleon assembly only. This nuclear equation rules the dependence of the potential both on the nucleon momentum and the nuclear density. The particular choice of the two-body forces or of the nucleon-nucleon phase shifts for

the evaluation of the potential energy of the nucleus, implies a specialization of the Cauchy problem, related to this equation, regardless of the saturation prescriptions of nuclear forces. It is shown that there exists a class of solutions of this nuclear equation which cannot be derived, in the first approximation of the perturbation method, from any of the known two-body potentials. This class of solutions, however, leads to the saturation of the nuclear binding energy and density as well as to the experimental value of the symmetry energy and to the correct behavior, in the low energy region, of the real and imaginary parts of the nuclear potential. A mathematical proof is given of the dependence of the potential inside the Fermi sphere on even powers of the nucleon momentum, as required by the invariance prescription of the potential with respect to time reflection. The linear dependence of the potential on the square of the nucleon momentum, and the so-called nucleon effective mass approximation, is discussed in the light of the correspondence principle, which has been used to describe the motion of a nucleon in nuclear matter. Finally, it is shown that the Johnson-Teller, Schiff-Thirring and Drell-Huang theories of nuclear saturation implicitly involve only particular solutions of the considered nuclear equations." (Author's summary) *A. Dalgarno* (Belfast)

7543:

Ikeda, Kiyomi; Kobayasi, Minoru; Marumori, Toshio; Nagata, Sinobu; and Shiozaki, Takanori. Foundation of deformed potential model for nuclear rotation. Progr. Theoret. Phys. **20** (1958), 960–970.

A separation of the nuclear Hamiltonian into rotational and other parts is achieved formally by a method which involves introducing certain redundant variables and applying successively three unitary transformations. The relationships of various methods of calculating moments of inertia corresponding to the deformed potential model are clarified. *A. Dalgarno* (Belfast)

7544:

Bethe, H. A.; and Goldstone, J. Effect of a repulsive core in the theory of complex nuclei. Proc. Roy. Soc. London. Ser. A **238** (1957), 551–567.

Brueckner's method is used to study the effect of an infinitely repulsive core on the properties of large nuclei, the exclusion principle being taken into account from the beginning. The effective mass approximation is discussed. *D. ter Haar* (Oxford)

7545:

Geilikman, B. T. On axially asymmetric nuclei. Soviet Physics. JETP **35(8)** (1959), 690–692 (989–991 Ž. Eksper. Teoret. Fiz.).

"The possibility of the existence of nuclei lacking axial symmetry is demonstrated within the framework of the generalized model." *Author's summary*

7546:

Królikowski, W. On the classification of elementary fermions. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 523–527.

A four-dimensional isobaric space is proposed in which baryons and leptons are made members of the same multiplets. A connection with a Fock space representation of a single field obeying Fermi statistics in isobaric space is suggested and this is used to limit the number of elementary particles that may exist. Room is found in the scheme for all known Fermions together with six unknown ones. Further assumptions are suggested to remove the latter if desired. *J. C. Polkinghorne* (Cambridge, England)

7547:

Mandelbrojt, Jacques. Vecteur d'état approché du nucléon habillé dans l'approximation du couplage intermédiaire. C. R. Acad. Sci. Paris **248** (1959), 530–533.

"On remplace l'équation aux vecteurs propres de l'approximation de couplage intermédiaire de Tomonaga par une condition nécessaire et suffisante pour que cette approximation donne exactement l'état fondamental du nucléon considéré comme source fixe. On obtient ainsi un système d'équations qu'on résout." *Résumé de l'auteur*

7548:

Merat, Parviz. Sur un formalisme spinoriel réel dans la théorie de l'électron. C. R. Acad. Sci. Paris **248** (1959), 533–535.

"On remplace les quatre composantes complexes du spineur de Dirac par huit composantes réelles. On introduit deux nouveaux quadrivecteurs potentiels à côté du quadrivecteur électromagnétique de l'électron sans modifier les formules de Møller et de Bhabha." *Résumé de l'auteur*

7549:

Butcher, J. C.; and Messel, H. Electron number distribution in electron-photon showers. Phys. Rev. (2) **112** (1958), 2096–2106.

This article is a preliminary report on a large scale machine calculation aiming at the tabulation of Φ, the probability of finding a given number of particles of a particular kind, above a given energy, at a certain depth below the cascade origin, the cascade having been originated by a given type of particle with a particular energy. The calculations are carried out for two cases. In the first, called approximation A, full screening cross sections are used for bremsstrahlung and pair production; other processes are completely neglected. In the second case accurate bremsstrahlung and pair production cross sections are used, and collision losses as well as Compton effects are taken into account. Scattering at low energies is, however, neglected. Monte Carlo methods are used on an electronic computer. Some numerical results are given in form of graphs and some features of the results are discussed. *M. J. Moravcsik* (Livermore, Calif.)

7550:

Petiau, Gérard. Sur un système d'équations d'ondes non linéaires décrivant un modèle particule-champ de spin 0 et $\hbar$. C. R. Acad. Sci. Paris **248** (1959), 1129–1132.

Étude d'un système d'équations aux dérivées partielles du premier ordre non linéaires décrivant simultanément un corpuscule et le champ créé par le corpuscule source, les dégénérescences linéaires de la théorie étant soit les équations d'ondes du corpuscule de spin 0, soit celles du corpuscule de spin $\hbar$. *Résumé de l'auteur*

7551:

Popov, V. S. The behavior of a particle of arbitrary spin in an external magnetic field. Soviet Physics. JETP **35(8)** (1959), 687–689 (985–988 Ž. Eksper. Teoret. Fiz.).

"A method of disentangling given by Feynman is used to solve the problem of the way the polarization of a particle possessing a magnetic moment changes in an external magnetic field." *Author's summary*

7552:

Milekhin, G. A. On the hydrodynamic theory of multiple production of particles. Soviet Physics. JETP 35(8) (1959), 682–684 (978–981 Ž. Eksper. Teoret. Fiz.).

"The problem of symmetries in the angular and energy distributions of secondary particles produced in nucleon-nucleus collisions or in collisions of two nuclei is considered on the basis of the hydrodynamic theory of multiple production of particles. It is shown that such symmetry appears in a certain special system of coordinates which is close to the center-of-mass system." *Author's summary*

7553:

Okabayashi, Takao. New interpretation of hyperonic charge and its generalization to leptons. Progr. Theoret. Phys. 20 (1958), 583–613.

Parity non-conservation is assumed for all weak interactions. Parity and strangeness conservation in strong interactions and non-conservation in weak interactions suggest some relation between the two conceptions. Hyperonic charge is used instead of strangeness. The pair (N, Ξ) is taken to belong to eigenvalues ± 1 of an operator (eventually $\zeta_2\gamma_5$) whose expectation value is hyperonic charge, and (Y, Z), two isotopic spin doublets from a combination of Λ, Σ particles, are taken as eigenstates, with hyperonic charge zero, of another operator (eventually $\zeta_3\gamma_5$) which anticommutes with $\zeta_2\gamma_5$: ζ_i are a new set of Pauli-type matrices. Modified equations for spinor fields ψ and $\gamma_5\psi$ are coupled to give three forms of equation such as

$$\{i(\alpha+\beta\zeta_1)\zeta_2\gamma_5\gamma_\mu\partial_\mu-m\}\Psi=0,$$

where $\Psi=\begin{pmatrix}\psi\\ \gamma_5\psi\end{pmatrix}$. These equations have, respectively, $\zeta_i\gamma_5$ $(i=1, 2, 3)$ as constants of motion, the corresponding Ψ belonging to eigenvalues $+1$. The hyperonic charge operator and the particle-antiparticle conjugation operator are assumed to anticommute, giving equal and opposite hyperonic charges to particle and anti-particle. The remainder of a very detailed paper considers the interactions between two spinor and one boson field π, K, the assignment of generalised hyperonic charge to leptons, the exclusion of unwanted processes, the relation of hyperonic charge to isotopic spin space, and other topics.

C. Strachan (Aberdeen)

7554:

Ivanenko, I. P. On the cascade theory of showers. Soviet Physics. JETP 35(8) (1959), 94–97 (132–136 Ž. Eksper. Teoret. Fiz.).

Ramakrishnan and Srinivasan [Proc. Indian Acad. Sci. Sect. A 44 (1956), 263–273; 45 (1957), 133–138; MR 19, 221] recently gave what they termed a "new approach to cascade theory", wherein they considered the average number of electrons with energies greater than E that are produced between absorber depths of 0 and t rather than the average number of electrons with energies greater than E at t. Ivanenko has considered the diffusion equations for the same problem when ionization losses by the electrons are taken into account. A solution of the diffusion equations has been obtained and an approximate numerical evaluation given for the average numbers.

H. Messel (Sydney)

7555:

Perel', V. I. Wave functions of many-electron systems. Soviet Physics. JETP 35(8) (1959), 476–479 (685–690 Ž. Eksper. Teoret. Fiz.).

"A general form has been derived for a wave function which is an eigenfunction of S^2 and S_z and satisfies the Pauli principle. An expression is given for the Schrödinger wave function of the system constructed from one-electron functions." *Author's summary*

7556:

Power, E. A.; and Shail, R. The interaction of light with neutral systems. Proc. Cambridge Philos. Soc. 55 (1959), 87–90.

A contact transformation is used to convert the interaction Hamiltonian of a system of particles and the electromagnetic potential into a gauge-invariant form involving multipole moments and the field strength. The new Hamiltonian is applied to the quantum theory of double refraction, and to optically active systems.

D. W. Sciama (London)

7557:

Landau, L. D. On the theory of the Fermi liquid. Soviet Physics. JETP 35(8) (1959), 70–74 (97–103 Ž. Eksper. Teoret. Fiz.).

This is an extension of previously published work [Soviet Physics, JETP 3 (1956), 920–935; 5 (1957), 101–108; MR 18, 975; 19, 786] on the quantum mechanics of many-body systems of Fermi particles. Field-theoretical techniques are used to find formal conclusions about the scattering matrix for 'quasi-particles" (excitations), especially in the limit at zero momentum transfer.

H. S. Green (Adelaide)

7558:

Klein, Abraham; and Zemach, Charles. Many-body problem in quantum field theory. Phys. Rev. (2) 108 (1957), 126–138.

Nucleon-nucleus and meson-nucleus collisions are treated by obtaining the relevant S-matrix using Green function techniques and a relevant adiabatic hypothesis.

D. ter Haar (Oxford)

7559:

Kirzhnits, D. A. Behavior of the distribution function of a many-particle system near the Fermi surface. Soviet Physics. JETP 34(7) (1958), 1116–1118 (1625–1628 Ž. Eksper. Teoret. Fiz.).

The Thomas-Fermi model gives fair approximations to properties of a many-electron system which depend on a smoothed average electron density: it is known, however, that in the region of the Fermi surface the model gives only poor results. This paper points out that the coordinates and momentum do not commute with the Hamiltonian, and derives the effects of the various commutators on the distribution function in the neighbourhood of the Fermi surface. These effects are considerable, and may be expected to give decidedly different results for those applications, such as exchange calculations, where a large contribution comes from the neighbourhood of the Fermi surface. *D. F. Mayers* (Oxford)

7560:

Anderson, P. W. Random-phase approximation in the theory of superconductivity. Phys. Rev. (2) 112 (1958), 1900–1916.

A theory is developed for the ground state of a superconductor which is consistent with basic postulates for the ground state function of the Bardeen-Cooper-Schrieffer theory [Phys. Rev. (2) 108 (1957), 1175–1204; MR 20 #2196] and of Bogoliubov's theory [Soviet Physics. JETP 34(7) (1958), 41–46; MR 20 #5670a; see p. 46] and at the same time can also handle collective effects (long range Coulomb interaction) in the Brueckner-Gell-Mann

approximation [Phys. Rev. (2) 106 (1957), 364–368; MR 19, 98]. The method is the generalization of the Bohm-Pines random-phase approximation [Phys. Rev. (2) 92 (1953), 609–625]. It is claimed that the theory: a) shows in a natural way why the restriction to a fixed number of electrons must be relaxed; b) can compute the correlation coefficients to superconductivity and shows that they are small; c) gives a good account of collective effects such as photons and higher bound states; d) gives a more physical picture of the superconductive state.

M. J. Moravcsik (Livermore, Calif).

RELATIVITY

See also 7290, 7487, 7536.

7561:

Klarsfeld, S. Sur le tenseur d'énergie-impulsion dans l'électrodynamique de Bopp. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. 6 (1957), no. 15, 59–62. (Romanian. French and Russian summaries)

The author shows that the present expression deduced for the symmetrical energy-momentum tensor in the relativistic electro-dynamic theory of Bopp differs from that deduced by Bopp [cf. Ann. Physik (5) 38 (1940), 345–384; 42 (1942), 573–608; MR 2, 336; 8, 124].

K. Bhagwandin (Oslo)

7562:

Klarsfeld, S. Les lois de conservation dans l'électrodynamique de Bopp. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. 6 (1957), no. 15, 63–66. (Romanian. French and Russian summaries)

The author applies Lorentz transformation (infinitesimal) methods to deduce a canonical Lagrange function for the energy-momentum tensor, with the corresponding conservation laws. He also shows that the energy-momentum tensor can be made symmetrical by means of the procedures of Belinfante [Physica 7 (1940), 449–474; MR 2, 336] and Rosenfeld [Acad. Roy. Belgique Cl. Sci. Mém. Coll. in 8° 18 (1940), no. 6; MR 2, 143]. The present result coincides with the one presented in the preceding review.

K. Bhagwandin (Oslo)

7563:

Kalitzin, Nikola St. Grundgleichungen der relativistischen Mechanik eines materiellen Punktes mit veränderlicher Masse. C. R. Acad. Bulgare Sci. 11 (1958), 185–188. (Russian summary)

The author continues with his study of the equations of motion (within the framework of the special theory of relativity) of a particle of variable rest-mass [same C. R. 7 (1954), no. 2 (1955), 9–12; Soviet Phys. JETP 28(1) (1955), 565–567 (Ž. Eksper. Teoret. Fiz. 631–632); MR 16, 1167; 17, 202]. A particle of rest-mass m, moving with velocity $u_i=dx_i/ds$ ($i=1, \cdots, 4$), coalesces with a particle of rest-mass dm' and velocity a_i. The resulting momentum p_i of the system is expressed in the form $c(m+dm)(u_i+du_i)$. The equations of motion which had been previously obtained (loc. cit) correspond to the case $dm=dm'$. The present paper is concerned with the case $dp_i=0$, which, according to the author, describes the motion of a rocket in empty space devoid of gravitational fields.

H. Rund (Durban)

7564:

Tulczyjew, W. On the motion of rotating bodies in the general theory of relativity. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 6 (1958), 645–651.

The equations of motion of two rotating bodies are stated without proof. In addition to the known "Einstein, Infeld, Hoffman terms" there are further terms involving quantities $\overset{1}{S}{}^{rs}$, $\overset{2}{S}{}^{rs}$ ($r, s=1, 2, 3$) which represent the internal angular momentum and satisfy $dS^{rs}/dt=0$. The present investigation considers the effect of these terms on the motion of the centre of gravity and on the relative motion of the bodies. By introducing a three-dimensional vector in terms of the S^{rs}, a number of relations are expressed as three-dimensional vector equations.

G. L. Clark (London)

7565:

Marder, L. Two bodies at rest in general relativity. Proc. Cambridge Philos. Soc. 55 (1959), 82–86.

"It is well known that there is no static axisymmetric two-body solution of Einstein's gravitational field equations, if it is assumed that the bodies are separated in a certain definite sense. In this paper it is shown, by the construction of a complete physically sensible model, that static two-body solutions do exist for systems in which one body is hollow and contains the other. The stability of the particular system described is briefly discussed." (Author's summary)

D. W. Sciama (London)

7566:

Thomas, L. H. General relativity and particle dynamics. Phys. Rev. (2) 112 (1958), 2129–2134.

"In order to adapt the Hamiltonian dynamics of a system of classical (non-quantum) particles to general relativity it seems necessary to generalise the transformations used from contact transformations of phase space to extended point transformations of a joint observer and phase space. When this is done the curvature of observer space is seen to arise from the physical interaction of particles without the introduction of new gravitational field variables.

In order to extend this theory to quantized particle and field theory it seems necessary to extend the unitary transformations of quantum mechanics in a similar manner, to linear transformations of matrices maintaining the trace of the product of two matrices, the Hermitian nature of a matrix, and the unit matrix. Such transformations do not preserve the phases of wave functions or the product of two matrices, but do preserve the characteristic values of matrices and the traces of the products of any numbers of matrices. That observers should be designated by q numbers, not c numbers, makes it difficult to interpret the theory except in the classical limit." (Author's summary.)

D. W. Sciama (London)

7567:

Lévy, Jacques. Astronomie de position et relativité générale. Cahiers de Phys. 12 (1958), 437–446.

The paper contains a general survey from the astronomical point of view of the present status of the rotation of the perihelion of Mercury and of the other planets, of the bending of light rays and of the gravitational red-shift.

G. C. McVittie (Urbana, Ill.)

7568:
Bonnor, W. B. Spherical gravitational waves. Philos. Trans. Roy. Soc. London. Ser. A **251** (1959), 233–271.

"The field of gravitational radiation emitted from two moving particles is investigated by means of general relativity. A method of approximation is used, and in the linear approximation retarded potentials corresponding to spherical gravitational waves are introduced. As is already known, the theory in this approximation predicts that energy is lost by the system. The field equations in the second, non-linear, approximation are then considered, and it is shown that the system loses an amount of gravitational mass precisely equal to the energy carried away by the spherical waves of the linear approximation. The result is established for a large class of particle motions, but it has not been possible to determine whether energy is lost in free gravitational motion under no external forces.

The main conclusion of this work is that, contrary to opinions frequently expressed, gravitational radiation has a real physical existence, and in particular, carries energy away from the sources." (Author's summary)

D. W. Sciama (London)

7569:
Bastin, E. W.; and Kilmister, C. W. The concept of order. III. General relativity as a technique for extrapolating over great distances. Proc. Cambridge Philos. Soc. **53** (1957), 462–472.

It is shown that a combination of the methods developed in the two previous papers in the series [same Proc. **50** (1954), 278–286; **51** (1955), 454–468; MR **15**, 760; **16**, 1179] allows the reformulation of general relativity as a technique for extrapolating physical theory to distant regions, whose accessibility to experiment is limited. General relativity is thus shown to be one of a number of extrapolation theories allowed by the spread theory of paper II. *Authors' summary*

7570:
Bastin, E. W.; and Kilmister, C. W. The concept of order. IV. Quantum mechanics. Proc. Cambridge Philos. Soc. **55** (1959), 66–81.

"This paper formulates quantum mechanics in terms of the sequences introduced in the first three papers of this series [see #7569 and references therein]. There are three sections: (I) Explanation of the existence of particles with an exhaustible set of attributes by means of these sequences; (II) demonstration that these particles are actually the elementary particles of quantum theory, by showing the isomorphism of a well-known form of quantum mechanics with the mathematics of sequences, except for the scale constant in the former; (III) explanation of the existence of the scale constant and the connexion of its finiteness with the finiteness of the velocity of light." (Authors' summary)

A. H. Taub (Urbana, Ill.)

7571:
Stephenson, G. Quadratic Lagrangians and general relativity. Nuovo Cimento (10) **9** (1958), 263–269.

The author considers three particular action principles with quadratic Lagrangians in general relativity theory, the fundamental tensor g_{ab} and components of linear connection $\Gamma_{ij}{}^{k}$ being varied independently of each other. After the variations have been made the $\Gamma_{ij}{}^{k}$ are then assumed to be Christoffel symbols. Of the three sets of equations derived, the first two are shown to be satisfied by any solution of the equations $R_{kl}=\lambda g_{kl}$, whilst the third set, C, is shown to be satisfied by the Schwarzschild metric. {The assumption $\Gamma_{ij}{}^{k}=\{ij, k\}$ makes the work rather trivial, and no new substantial results are derived. Also, as regards C, one easily obtains the much stronger result that this set is also satisfied by any solution of $R_{kl}=\lambda g_{kl}$, if one uses an identity of the reviewer's which occurs in a paper quoted by the author [J. London Math. Soc. **26** (1951), 150–152; MR **12**, 859].}

H. A. Buchdahl (Princeton, N.J.)

7572:
Buchdahl, H. A. On the compatibility of relativistic wave equations for particles of higher spin in the presence of a gravitational field. Nuovo Cimento (10) **10** (1958), 96–103. (Italian summary)

The author has generalized the standard free-field equations for particles of spin 3/2 by Fierz and Pauli [Fierz, Helv. Phys. Acta **12** (1939), 3–37; Fierz and Pauli, Proc. Roy. Soc. London Ser. A **173** (1939), 211–232; MR **1**, 190] for a curved space and finds that these equations possess an internal inconsistency unless the metric of the underlying space itself satisfies Einstein's field equations for the pure gravitational field without sources. As the author himself points out, his result becomes physically unacceptable if the reaction of the particle field itself on the gravitational field is taken properly into account. The alternative, to obtain field equations from a variational principle, introducing if necessary auxiliary spin vectors, is rejected by the author as "rather artificial". He offers no detailed discussion of that alternative. *P. G. Bergmann* (New York, N.Y.)

7573:
Teodorescu, Ion D. Sur les équations électromagnétiques du champ électrogravitique dans la théorie unitaire nonholonome. An. Univ. "C. I. Parhon" București. Ser. Ști. Nat. **6** (1957) no. 15, 25–41. (Romanian. French and Russian summaries)

The author studies the properties of the non-holonomic space $V^4{}_5$ from the point of view of the non-holonomic theory [e.g., cf. Vranceanu, J. Phys. Radium (7) **7** (1936), 514–526; Ann. Mat. Pura Appl. (4) **6** (1929), 9–43].

He obtains the electro-magnetic equations, in the gravitational field in a five-dimensional space, which are the generalized covariant equations of Maxwell in the theory of relativity. He also deduces this set of equations by means of the Lagrangian variational principle.

K. Bhagwandin (Oslo)

7574:
Glasko, V. B.; Leriust, F.; Terletskii, Ia. P.; and Shushurin, S. F. An investigation of particle-like solutions of a nonlinear scalar field equation. Soviet Physics. JETP **35**(8) (1959), 312–315 (452–457 Ž. Eksper. Teoret. Fiz.).

The mass-spectrum of a complex scalar field equation containing a cubic nonlinear term is investigated for a range of parameters using an electronic computer. Particle-like solutions are defined by behavior at the origin and infinity, and mass is defined through the time dependence of the solutions. Otherwise no physical criteria are introduced to discuss the solutions of the very special equation considered.

S. Bludman (Berkeley, Calif.)

7575:
Urbakh, V. Iu. A nonlinear theory of vector fields. Soviet Physics. JETP **35**(8) (1959), 143–148 (208–215 Ž. Eksper. Teoret. Fiz.).

The author constructs a non-linear generalization of the

classical theory of the electromagnetic field in vacuum. The theory is formulated in terms of spinor-tensors. The space-time metric spinor-tensor is given by $b_{ik}=\frac{1}{2}(\gamma_i\gamma_k+\gamma_k\gamma_i)$. (2) $\gamma_i=\alpha_i+h_i$, where the α_i are Dirac matrices and $h_i=(e/m_0c^2)A_i$, where A_i is the classical electromagnetic potential. Thus, in general, the space possesses curvature fixed by the electromagnetic potential. The electromagnetic field is given by $\Gamma_{ik}=\partial_i\gamma_k-\partial_k\gamma_i$, and the field equations are the Lagrange equations of a variational principle with $\mathscr{L}=\text{const}\, b^{im}b^{jn}\Gamma_{ij}\Gamma_{mn}$, analogous to classical theory. However, the field equations for h_i are non-linear since **b** depends on **h** through (1) and (2). The equation of a static, spherically symmetric electric potential is obtained, and an approximate solution is claimed to have been determined. The solution departs sensibly from the classical solution $\varphi=\text{const}/r$ at a distance $r\sim e^2/m_0c^2$. The solution is singular at $r=0$, but the total field energy is finite and given approximately by $m\approx m_0/\sqrt{22}$.

R. A. Toupin (Washington, D.C.)

7576:

Costa de Beauregard, Olivier. Sur la théorie de l'inertie de D. W. Sciama et D. Park et sur la relation de la "longueur élémentaire" λ aux constantes universelles c, h, G. C. R. Acad. Sci. Paris **247** (1958), 2101–2103.

"On propose un système d'unités tel que $\lambda=c\tau$, $c\lambda\mu=h$, $\mu G_{(\lambda,\mu)}=c^2\lambda$ (λ, τ, μ, quanta de longueur, temps, masse; c, h, constantes de la relativité restreinte et des quanta; $G_{(\lambda,\mu)}$ définition microphysique de la constante de gravitation). La loi d'inertie devient alors $F=\text{Im}\gamma$, avec $I=2G_{(\lambda,\mu)}M/\pi c^2R$; M, masse totale et R, rayon du cosmos sphérique d'Einstein." (Author's summary.)

D. W. Sciama (London)

7577:

Costa de Beauregard, Olivier. L'hypothèse de l'effet gravitationnel de spin. C. R. Acad. Sci. Paris **247** (1958), 1092–1094.

The author outlines a quasi-Minkowskian version of the theory proposed by the reviewer [Proc. Cambridge Philos. Soc. **54** (1958), 72–80; MR **20** #727], according to which spinning masses produce a non-symmetrical gravitational field. The author's equation of motion for a spinless test-particle in such a field differs from the reviewer's, but whereas the latter is deduced from the field equations, the former appears to be assumed ad hoc.

D. W. Sciama (London)

ASTRONOMY

See also 7488, 7493.

7578:

Meffroy, Jean. Sur l'origine du terme séculaire pur de la perturbation du troisième ordre des grands axes. C. R. Acad. Sci. Paris **248** (1959), 1294–1297, 1773–1776.

In an earlier note [C. R. Acad. Sci. Paris **240** (1955), 1054–1056; MR **16**, 1160] the same author gave a proof of the existence of secular terms in the major axes in the mutual perturbations of two planets, provided that the eccentricities and inclinations are sufficiently small. These two notes establish in detail the origin of these purely secular terms as they arise (a) from the secular part, and (b) from the periodic part of the disturbing function.

D. Brouwer (New Haven, Conn.)

7579:

Novoselov, V. S. On the problem of the movement of two bodies with variable masses. Vestnik Leningrad. Univ. **12** (1957), no. 13, 129–131. (Russian. English summary)

The author considers the two-body problem for particles of variable mass; a particular form of the differential equations is assumed, on the basis of a physical model. A necessary and sufficient condition is obtained, in terms of the variation of mass, that the distance between the particles can become infinite as time becomes infinite.

W. Kaplan (Ann Arbor, Mich.)

7580:

Priester, Wolfgang. Zur Statistik der Radioquellen in der relativistischen Kosmologie. Z. Astrophys. **46** (1958), 179–202.

The cumulative number, N, of class II radio-sources counted to limiting flux-density S is found empirically to follow the rule that N is proportional to $S^{-(3+\mu)/2}$, where μ is $\pm.2$. The author considers the three homogeneous universes in which the pressure is zero and the cosmical constant is zero. He assumes that class II radio-sources are colliding galaxies and shows that the number of such sources in a unit volume should be proportional to $(1+\delta)^6$, where δ is the observed red-shift in the light from an optical source located at the unit volume in question. The (N, S) formula calculated from this hypothesis and the theory of the model universes is shown to be in fair agreement with the empirical rule.

G. C. McVittie (Urbana, Ill.)

GEOPHYSICS

See also 7433, 7474.

7581:

Klušin, I. G. On the extraction of geophysical anomalies less than the mean-square error of measurement. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1959**, 189–196. (Russian)

An application of the techniques of least squares filtering to the extraction of geophysical data in the presence of strong noise. *L. A. Zadeh* (New York, N.Y.)

7582:

Ivakin, B. N. Elastic media with non-ideal inertia and their models. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1959**, 210–220. (Russian)

In contrast to two of the author's previous papers in which the problem of media and their models with non-ideal elasticities are treated, in this paper one considers the problem of elastic media with non-ideal inertia; within such media the masses are coupled with absorbable elements (e.g. dampers) and consequently absorb the elastic energy.

The analogy between the equations of motion of media in the first case and in the case of ideal elastic media with non-ideal inertia (and their models) is shown. The analogy between the law of dispersion of velocities and the absorption of waves in these different media is also pointed out. A discussion is given of different wave strengths caused by elastic hysteresis. In this case the equations of motion of a medium with elastic hysteresis is written in Sokolov-Skrabin's form and the operational form is used. The mechanical and electrical models for the case

of an absorbable medium with inertia hysteresis and with elastic and inertia hysteresis (real seismic medium) are presented. *D. P. Rašković* (Belgrade)

7583:

Löbel, Paul. Vektorielle Ausgleichung trigonometrischer Punkte. Z. Vermessungswesen **83** (1958), 466–482.

The application of vector methods to survey adjustments is demonstrated for the insertions of a single station and two stations into a trigonometric net. A numerical example is worked out in detail for each case. The amount of computation involved is much less than for the conventional coordinate adjustment, as can be seen by comparing the author's tabular forms with, e.g., those of W. Grossmann [*Grundzüge der Ausgleichungsrechnung nach der Methode der kleinsten Quadratic nebst Anwendung in der Geodäsie*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1953; MR **15**, 650; pp. 128–137].

The author asserts that he has, in conjunction with K. Friedrich, fully developed the application of vector methods to all phases of survey adjustments, and that he hopes to publish further results later.

B. Chovitz (Washington, D.C.)

7584:

Bodemüller, H. Berechnung langer geodätischer Linien. Z. Vermessungswesen **83** (1958), 453–466.

This is an expository article, discussing various methods of computing the forward and inverse solutions of geodetic lines on the surface of the earth considered as an ellipsoid of revolution. The article concentrates on long lines, i.e., of length 1000 kilometers or greater. The material consists essentially of a synopsis of the author's previous publications [Deutsche Geodätische Kommission, Bayer. Akad. Wiss. Reihe B: Angew. Geodäsie, no. 13 (1954); no. 18 (1955)] which expound in detail the classical methods of Bessel and Helmert; and also brings the state of the art up-to-date by mentioning modern methods for the inverse solution by Levallois and Dupuy [*Note sur le calcul des grandes géodesiques*, Imprimerie de l'Institut Géographique National, Paris, 1952] and E. M. Sodano [Bull. Géodésique N.S. no. **48**, 13–25 (1958)]. Qualitative and very general remarks on these methods with respect to accuracy and ease of computation are made. *B. Chovitz* (Washington, D.C.)

OPERATIONS RESEARCH AND ECONOMETRICS

See also 7376.

7585:

Zoroa Terol, Procopio. A decision sequential problem in economics. Trabajos Estadist. **9** (1958), 103–110. (Spanish. English summary)

The author discusses the problem of a merchant who can sell an item in any one of m time periods, profit being a function of the time of sale. Profit is a random variable whose distribution may be estimated from past experience. A sequential decision rule is given for a special case. *H. S. Houthakker* (Cambridge, Mass.)

7586:

Mitten, L. G. Sequencing n jobs on two machines with arbitrary time lags. Management Sci. **5** (1959), 293–298.

The author considers a sequencing problem involving running n jobs first on machine I and then on machine II. Running times are given for each machine, and there is a time lag associated with the ith job, requiring that the second machine not be used until d_i time units after the first machine has finished its part. The author determines the optimal sequence, using an extension of the method given by Johnson [Naval Res. Logist. Quart. **1** (1954), 61–68]. *R. Bellman* (Santa Monica, Calif.)

7587:

Johnson, S. M. Discussion: sequencing n jobs on two machines with arbitrary time lags. Management Sci. **5** (1959), 299–303.

The author presents a simpler proof of Mitten's result, showing that it follows as a special case of a three-machine problem already considered.

R. Bellman (Santa Monica, Calif.)

7588:

Greenberg, Harold. An analysis of traffic flow. Operations Res. **7** (1959), 79–85.

In a simple wave in gas dynamics, the velocity u and the density are related by $u-\int \rho^{-1}c d\rho=$constant, where c is the sound speed. The author proposes this relation, with $c=$constant, for the velocity-density relation in traffic flow. Quite apart from questions of why formulas from gas dynamics should apply more than qualitatively to traffic flow, the above Riemann relation is only meaningful in an unsteady wave motion, whereas the required relation in traffic flow refers essentially to steady state conditions. Reference is made to the paper by M. J. Lighthill and G. B. Whitham [Proc. Roy. Soc. London Ser. A **229** (1955), 317–345; MR **17**, 310], and it is claimed that "the fluid dynamic analogy is developed further". In fact, the work cited was a development coming from work on simple waves in gas dynamics but with due attention to important differences; thus, it would appear that the paper under review is a step backwards.

For suitable choice of c, the relation $u=c\log \rho/\rho_0$ is found to fit certain observational traffic data and may have value as an empirical formula.

G. B. Whitham (New York, N.Y.)

7589:

Bellman, Richard; and Dreyfus, Stuart. On the computational solution of dynamic programming processes, a bottleneck processes arising in the study of interdependent industries. J. Operations Res. Soc. Japan **2** (1958), 1–10.

The authors consider a special sort of time-phased linear programming problem with many variables and constraints, but with a constraint matrix which is block-diagonal, and shows how a dynamic-programming formulation permits a recursive computation with reduced dimensionality. *R. Solow* (Cambridge, Mass.)

7590:

Wagner, Harvey M. The dual simplex algorithm for bounded variables. Naval Res. Logist. Quart. **5** (1958), 257–261.

This paper gives the algorithm for solving by the dual simplex method the linear programming problem max $c'\lambda$ subject to $A\lambda\leqq b$; $0\leqq\lambda\leqq u$.

The simplex method starts with a feasible solution and works towards an optimal feasible solution, whilst the dual method is that which starts with an infeasible optimal solution and works towards a feasible optimal solution.

When the variables, λ_i, are bounded above (by u_i), the

direct algorithm becomes much more complicated, but the dual algorithm can be kept in a simple form by the neat device of replacing, after each cycle of the algorithm, any variable λ_i which has exceeded its upper bound by its complementary variable λ_i', where $\lambda_i+\lambda_i'=u_i$. This involves only a set of sign changes, relabellings, and a trivial numerical substitution, and returns the tableau to standard form. These changes do not destroy the convergence of the algorithm. The initial tableau is set up by replacing all variables with positive coefficients c_i in the optimising form by their complementary variables.

Martin Fieldhouse (Cambridge, England)

7591:

Schupack, Mark B. Economic lot sizes with seasonal demand. Operations Res. 7 (1959), 45–57.

If we assume 1) future demand is known with certainty, 2) demand is constant over time, and 3) no shortages are allowed, then we have

$$V=\tfrac{1}{2}QMC+RS/Q,$$

where V=total variable cost per unit time, Q=size of lot purchased, M=carrying cost per unit time as per cent of unit cost, C=unit cost of the item, R=demand per unit time, and S=cost of placing one order. By differentiation the minimum of Q is $(2RS/MC)^{\frac{1}{2}}$. In this paper the author considers the case when assumption 2) does not hold, that is, when demand is seasonal and not constant over time.

The length of the demand cycle, which is usually one year, is identified with the point 2π, on a horizon beyond which the item will not be kept in stock. If Q_n is the size of the order for new stock and X_n the time to reorder, then the area under the curve bounded by $x=X_{n-1}$ on the left, from (Q_n, X_{n-1}) falling to $(0, X_n)$, when divided by the time X_n-X_{n-1}, is I_{aj}, the average inventory in stock. The methods of harmonic analysis are used, and the demand function assumes the form

$$D=a_0+\sum_{j=1}^{\frac{1}{2}N} a_j \cos_j(X-\alpha)+\sum_{i=1}^{\frac{1}{2}N-1} b_i \sin_i(X-\alpha),$$

where X is the time in radians, α is a phase angle to fit the curve to the middle of each planning period, and N is the number of planning periods, within the horizon cycle. Then the amount demanded from time X_f to time X is $\int_{x_f}^{x} Ddx$ and the inventory held is

$$I_i=\int_{x_f}^{x_n}\left\{\int_{x_f}^{x_n} Ddx-\int_{x_f}^{x} Ddx\right\}dx.$$

For n orders per cycle we have

$$I_{tn}=\left\{\int_0^{x_1}+\int_{x_1}^{x_2}+\int_{x_2}^{x_3}+\cdots+\int_{x_{n-1}}^{2\pi}\right\}Idx.$$

The point at which the minimum total cost occurs is reached by the empirical method of calculating in succession $I_{tn}MC+nS$, for $n=1, 2, \cdots, N$, and picking out the numerical minimum from this sequence. Two examples are worked out in detail. The possible generalizations of the ideas in this short paper hold the promise of an interesting new field of study in the analysis of demand.

L. J. Slater (Cambridge, England)

7592:

Weiss, Herbert K. Some differential games of tactical interest and the value of a supporting weapon system. Operations Res. 7 (1959), 180–196.

The author solves a four-parameter family of differential games interpreted as battles between two forces consisting of infantry with air support, and discusses in some detail the implications concerning cost and composition of forces. The method of getting the solutions is intelligent guessing followed by a proof, which seems to be still the best method known in this field.

J. Isbell (Seattle, Wash.)

7593:

★Burger, Ewald. Einführung in die Theorie der Spiele: Mit Anwendungsbeispielen, insbesondere aus Wirtschaftslehre und Soziologie. Walter de Gruyter & Co., Berlin, 1959. 169 pp.

The most remarkable feature of this in many ways noteworthy book is the large amount of material which the author has been able to include in its 169 pages. At the time of this writing nothing comparable exists in book form in any language.

The book treats game theory as a branch of applied mathematics, which is what it is, and accordingly great emphasis is placed on the applications, primarily to economics. Actually, many of these applications consist of topics which do not fall directly in the game category. These include quite thorough treatments of linear programming, exchange equilibrium and the production models of von Neumann and Wald.

The book is divided into four chapters. The first, "Der allgemeine Spielbegriff" leads up to the fundamental definitions of game theory by a series of eight examples of increasing complexity. These examples, all of which are "solved" in subsequent parts of the book, include finite, infinite, 2-person, n-person, zero-sum and non-zero-sum games and range from "Scissors, paper, rock" to poker to games of economic competition and military strategy. These examples, along with many others which are introduced in later chapters, constitute one of the book's important assets.

Chapter II on non-cooperative theory treats general n-person theory and has as its principal theme the concept of equilibrium point. (It is interesting that the author has chosen to go from the general to the particular so that the minimax theorem for matrix games is first proved as a special case of a general equilibrium point theorem.) The first main theorem of the chapter is the Zermelo-von Neumann-Kuhn theorem on the existence of equilibrium points for finite games with perfect information. It is then shown by means of the example of Gale and Stewart that the theorem no longer holds for infinite games. These are essentially the only results in the book on games in extensive form. The second main result of the chapter is the equilibrium point theorem of Nikaido-Isoda from which the Nash theorem is derived as a special case. From the general theorem the author proves an equilibrium point theorem for continuous games in which the mixed strategies are measures on a real line interval. The final section treats a price equilibrium model following Gale and using Debreu's notion of a generalized game.

The third and longest chapter of the book is devoted to the theory of two-person zero-sum games. Included are proofs of the minimax theorem for games with perfect information (von Neumann), matrix games (the von Neumann "Hauptsatz"), convex games (Bohnenblust-Karlin-Shapley), continuous games (Ville), and "measurable" games (Wald). There follows a section on matrix games whose principal theorem is the Shapley-Snow theorem characterizing basic solutions. Next comes a 22-page section presenting the entire theory of linear programming, including a proof of convergence of the generalized simplex method of Dantzig, Orden, and Wolfe (this was the one place noticed by the reviewer where the appropriate reference

[G. B. Dantzig, A. Orden, and P. Wolfe, Pacific J. Math. 5 (1955), 183–195; MR 16, 1045] seemed to be missing}. As related applications the next section treats the expanding linear model of von Neumann and Kuhn's version of the Wald-Walras production equilibrium theorem. The final section, concerned with infinite games, proves the minimax theorem for certain almost continuous games, including Karlin's theorem on functions discontinuous on the diagonal of the unit square.

The final chapter is concerned with the cooperative theories of von Neumann and Morgenstern and of Shapley. Besides general theorems of the von Neumann theory the solution concept is illustrated by exhibiting solutions for the case of Shapley's market games and Bott's symmetric (n, k)-majority games. The final section defines, proves existence of, and illustrates by example the Shapley value.

An appendix is devoted to giving proofs of the Brouwer and Kakutani fixed-point theorems via the Sperner lemma.

The book will be indispensable to anyone giving an advanced course in game theory, although it probably would not be usable as the sole text in such a course, principally because it contains no exercises. As a reference work on the subject it strikes the reviewer as the finest to date. *D. Gale* (Providence, R.I.)

7594:

Fürst, Dario. Un'applicazione della teoria dei giuochi. Archimede 10 (1958), 211–223.

A bank has 3 branches, one of which is going to be attacked by bandits. The bank guard is strong enough to defend one branch only. The author writes down the matrix for this 3×3 game and finds the solution.

W. H. Fleming (Providence, R.I.)

7595:

Shapiro, Harold N. Note on a computation method in the theory of games. Comm. Pure Appl. Math. 11 (1958), 587–593.

Julia Robinson [Ann. of Math. (2) 54 (1951), 296–301; MR 13, 261] showed the convergence of the method of "fictitious play" for obtaining the von Neumann value of a finite matrix game. The author shows that the same proof also yields an upper bound on the rate of convergence: After n steps of the procedure, the value of an r-by-s matrix game is obtained to within $O(n^{-1/(rs-2)})$, and the value of a symmetric r-by-r game to within $O(n^{-1/(r-1)})$.

P. Wolfe (Santa Monica, Calif.)

INFORMATION AND COMMUNICATION THEORY

See 7337, 7362, 7581.

CONTROL SYSTEMS

7596:

Boyarinov, V. S.; and Leonov, N. N. On the theory of a relay servo-system. Avtomat. i Telemeh. 19 (1958), 114–134. (Russian)

The authors present a detailed study of the trajectories of a second order relay system characterized by

$$(1)\qquad \ddot{x}+2h\dot{x}+x=F(\psi),\quad \psi=x+k\dot{x},$$

where k is a constant, h is a positive constant, and the relay characteristic F is of a rectangular type admitting as a special case a characteristic with dead zone. It is pointed out that the analysis of a similar type of system by Flügge-Lotz and Klotter [*Discontinuous automatic control*, Princeton Univ. Press, Princeton, N.J., 1953; MR 15, 529] is incomplete and contains several errors, one of which is the assertion that systems of type (1) have but one limit cycle, whereas in fact they may have two and sometimes three limit cycles.

L. A. Zadeh (New York, N.Y.)

7597:

Kochen, M. Extension of Moore-Shannon model for relay circuits. IBM J. Res. Develop. 3 (1959), 169–186.

The author defines a numerical reliability score for indicating the faithfulness with which variables redundantly represented by unreliable relays of the sort discussed by the reviewer and Shannon [J. Franklin Inst. 262 (1956), 191–208, 281–297; MR 18, 549] can be transformed by a circuit to give a Boolean function of these variables. He proposes particular circuits for "and", "or", and "exclusive-or", and obtains estimates of his reliability score for these circuits, contrasting them with the scores for certain alternative circuits which he directly modifies from the paper cited. He states: "The redundancies required to achieve a specified increase in reliability, although considerably smaller than for alternative methods, are still enormous."

E. F. Moore (Murray Hill, N.J.)

7598:

Lupanov, O. B. The synthesis of contact circuits. Dokl. Akad. Nauk SSSR (N.S.) 119 (1958), 23–26. (Russian)

Shannon has shown [Bell System Tech. J. 28 (1949), 59–98; MR 10, 671] that, for arbitrary $\varepsilon>0$ and $n>n(\varepsilon)$, the minimum number of contacts needed for the realization of any n-place Boolean function is bounded by $(1-\varepsilon)2^n/n<L(n)<2^{n+2}/n$. The author exhibits a method of synthesis of such functions which requires, asymptotically, $2^n/n$ contacts. *L. A. Zadeh* (New York, N.Y.)

7599:

Huzino, Seiiti. On some sequential machines and experiments. Mem. Fac. Sci. Kyusyu Univ. Ser. A 12 (1958), 136–158.

This paper consists of a large collection of definitions and results about machines in the sense of E. F. Moore [see Automata Studies, pp. 129–153, Princeton University Press, Princeton, N.J., 1956; MR 17, 1140]. Almost all the propositions follow immediately from the definitions. The author's main result, theorem 6.2, is a generalization of theorem 4 of Moore (cited above). Theorem 6.2 states that to each machine M which is the direct sum of strongly connected machines there is a unique minimal state machine S which is indistinguishable from M. Furthermore, S is fundamental (i.e., each two distinct states in S are distinguishable) and is the direct sum of strongly connected machines. {Reviewer's remark: Definition 6.1 is not strong enough to make proposition 6.1 true. In definition 6.1, π^{-1} should depend on π and x.}

S. Ginsburg (Hawthorne, Calif.)

7600:

Huzino, Seiiti. Reduction theorems on sequential machines. Mem. Fac. Sci. Kyusyu Univ. Ser. A 12 (1958), 159–179.

Using the terminology introduced in the paper above, the author considers the reduction of superfluous states

in a machine. In particular, boundary operators on a machine are introduced and various properties about them studied. (A boundary operator with respect to a particular state x_i in a machine M yields the machine obtained by removing x_i from M and changing those state transitions which, in M, led to x_i to the identity transition.)

{The reviewer finds the notation confusing. As the reviewer interprets their statement, propositions 1.3 and 3.1 and corollary 3.12 are incorrect (simple counter-examples exist). Theorem 4.1 may be false since it depends on proposition 3.1. Finally, proposition 1.1 is false as is, but becomes true if $n \geqq 2$ is changed to $n \geqq 3$.}

S. Ginsburg (Hawthorne, Calif.)

HISTORY AND BIOGRAPHY

See also 7410.

7601:

Nyien, Tun-gyeh. Li Yen's "Collected papers on history of Chinese mathematics". Advancement in Math. 3 (1957), 335–339. (Chinese)

7602:

Lusis, A. Ya. The development of mathematics in Soviet Latvia in the last decade. Latvijas Valsts Univ. Zinātn. Raksti **20** (1958), no. 3, 5–20. (Russian. Latvian summary)

A description, with complete bibliography, under 14 headings, beginning with mathematical logic and ending with the history of mathematics.

7603:

Errera, A. Sur les travaux de M. L. A. Antoine. Bull. Soc. Math. Belg. **9** (1957), 50–58.

Biography, and discussion of Antoine's mathematical contributions.

7604:

Liber, A. E.; Penzov, Yu. E.; and Raševskiĭ, P. K. Viktor Vladimirovič Vagner (on his fiftieth birthday). Uspehi Mat. Nauk **13** (1958), no. 6(84), 221–227. (1 plate) (Russian)

A scientific biography, with a photograph and a bibliography of 62 entries.

7605:

Gel'fond, A. O.; Leont'ev, A. F.; and Šabat, B. V. Alekseĭ Ivanovič Markuševič (on his fiftieth birthday). Uspehi Mat. Nauk **13** (1958), no. 6(84), 213–220. (1 plate) (Russian)

A short scientific biography, with a photograph and a bibliography of 83 entries.

GENERAL

See also 7378.

7606:

★Wilson, Edwin Bidwell. Advanced calculus. A text upon select parts of differential calculus, differential equations, integral calculus, theory of functions, with numerous exercises. Dover Publications, Inc., New York, 1959. ix+566 pp. $2.45.

An unaltered republication of the first edition published by Ginn and Co. [Boston-New York, 1912].

7607:

★Sawyer, W. W. A concrete approach to abstract algebra. W. H. Freeman and Co., San Francisco, 1959. iii+233 pp. $1.25.

This book was written for an institute for high school teachers. Fundamental facts regarding abstract fields and vector spaces are introduced by means of many familiar examples.

7608:

★Mathematisches Wörterbuch. Russisch-Deutsch. Mit einer kurzen Grammatik. Mathematical dictionary. Russian-English. With a short grammar. VEB Deutscher Verlag der Wissenschaften, Berlin, 1959. 244 pp. DM 9.80.

The short (23 pages) grammatical sketch of the Russian language is given first in German, then in English. The dictionary (about 5000 entries) is in three columns, Russian-German-English, under the Russian alphabetical order; applied mathematics and related fields are not included. A two-page list is given of the Russian names of the commonest mathematical symbols. The dictionary (conveniently thumb-indexed) includes, in addition to mathematical technical terms, the words of everyday language used in mathematical texts, and also the names of mathematicians (usually with very brief biographical data) which are most likely to be obscured by transliteration. The book will be useful to mathematicians with even a very small knowledge of Russian.

7609:

★Insights into modern mathematics. Twenty-Third Yearbook. The National Council of Teachers of Mathematics, Washington, D. C., 1957. viii+440 pp.

Eleven expository articles. The authors and titles are: J. Niven, Concept of number; E. J. McShane, Operating with sets; C. B. Allendoerfer, Deductive methods in mathematics; S. MacLane, Algebra; W. Prenowitz, Geometric vector and analysis and the concept of vector space; J. F. Randolph, Limits; R. E. Langer, Functions; S. H. Gould, Origins and development of concepts of geometry; R. H. Bing, Point set topology; H. Robbins, Theory of probability; C. B. Tompkins, Computing machines and automatic decisions. There is an introduction by C. V. Newsom and a concluding chapter by B. E. Meserve on implications for the mathematics curriculum.

BIBLIOGRAPHICAL NOTES

Institut des Hautes Études Scientifiques. Publications Mathématiques. No. 1, dated 1959 is a monograph (23 pp., 300 francs) by G. E. Wall with the title "The structure of a unitary factor group". Subsequent publications in the series will appear non-periodically in separate issues at different prices. Each issue may be bought separately. Special terms will be offered to yearly subscribers. Subscriptions may be addressed to: Presses Universitaires de France, 108, Boulevard Saint-Germain, Paris (6e).

Izvestiya Akademii Nauk SSSR. Otdelenie Tehničeskih Nauk. Metallurgia i Toplivo. (Metallurgy and Fuel.) Issue no. 1 is dated January–February 1959.

The Journal of the Australian Mathematical Society. Vol. 1, part 1 is dated August 1959.

For the present the Journal will be published half-yearly, four half-yearly numbers constituting a volume. The price per volume is £ 6.0.0 (Australian). Subscriptions should be sent to the treasurer of the Society (C.S. Davis, University of Queensland, Brisbane, Queensland, Australia).

Journal of Fluid Mechanics, published by the Cambridge University Press. Volumes 1 (May–Dec. 1956) through 4 (May–Nov. 1958) contained six parts each. Two volumes, 5 and 6, four parts each, will be published in 1959. The price per volume remains at £5.10 s. = $18.50.

Soviet Physics. Solid State.

In January, 1959, the "Soviet Journal of Technical Physics Section B" became simply the "Soviet Journal of Technical Physics", while Section A was changed to "Solid State Physics" (Fizika Tverdogo Tela). The American Institute of Physics, which has been translating the Journal of Technical Physics, will also translate the new journal. The journal and its translation both begin with the January 1959 issues. One volume of 12 issues will be published annually; the numbering will coincide with that of the original. The annual subscription price for the translation is $55.00 in the United States and Canada, $59.00 elsewhere; for libraries of non-profit degree-granting institutions, $25.00 and $29.00 respectively; single issues $8.00. Subscriptions should be addressed to the American Institute of Physics, 335 East 45th St., New York 17, N.Y.

Synthèse: An international quarterly for the logical and the psychological study of the foundations of science. Beginning with vol. XI, no. 1, March 1959 the journal will appear regularly as a quarterly with a new format. The contents also will be adapted to a more "multi-disciplinary approach". Annual subscription, to be addressed to D. Reidel Publishing Co., Dordrecht, Holland, is $7.50, 56 s. or f 28. The price of a single copy is $2.50, 20 s. or f 10.

Technometrics. Published quarterly in February, May, August and November by the Technometrics Management Committee for the American Society for Quality Control and the American Statistical Association. Editorial Office: 167 Nassau St., Princeton, N.J. The purpose of Technometrics is to contribute to the development and use of statistical methods in the physical, chemical and engineering sciences. Vol., 1, no. 1 is dated February 1959. The annual subscription rate is $8.00 a year ($6.00 a year for members of either of the two sponsoring societies).

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